Projective Geometric Algebra: A Swiss army knife for doing Cayley-Klein geometry

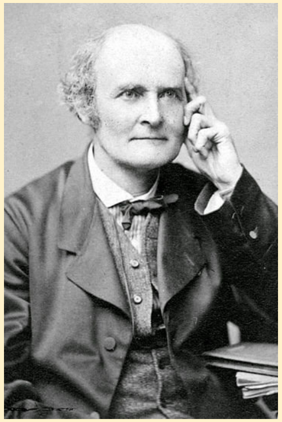
Charles Gunn

Sept. 18, 2019 at ICERM, Providence

Full-featured slides available at: https://slides.com/skydog23/icerm2019.

Check for updates incorporating new ideas inspired by giving the talk.

This first slide will indicate whether update has occurred.



What is Cayley-Klein geometry?

Example: Given a conic section $\mathbf{Q}$ in $\mathbb{R}P^2$.

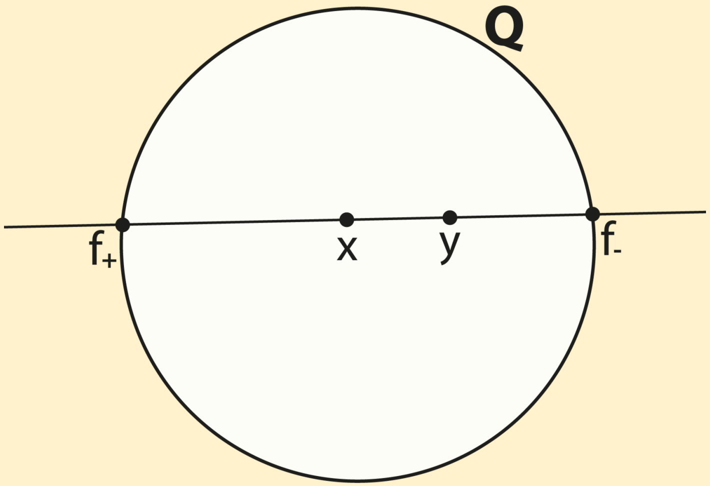
For two points x and y "inside" $\mathbf{Q}$, define

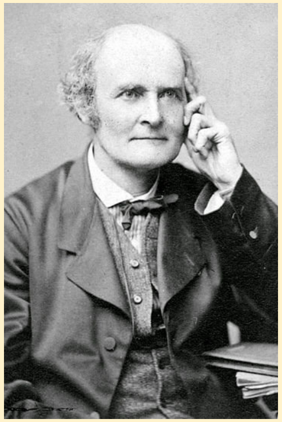
$$d(a, b) = log(CR(f_+, f_-; x, y))$$

where f_+, f_- are the intersections of the line xy with $\mathbf{Q}$ and CR is the cross ratio.

CR is invariant under projectivities

$\Rightarrow$ *d* is a distance function and the white region is a model for hyperbolic plane $\mathbf{H}^2$.



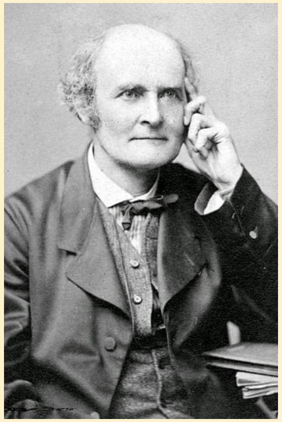


What is Cayley-Klein geometry?

SIGNATURE of Quadratic Form

Example: $(++-0) = (2, 1, 1)$

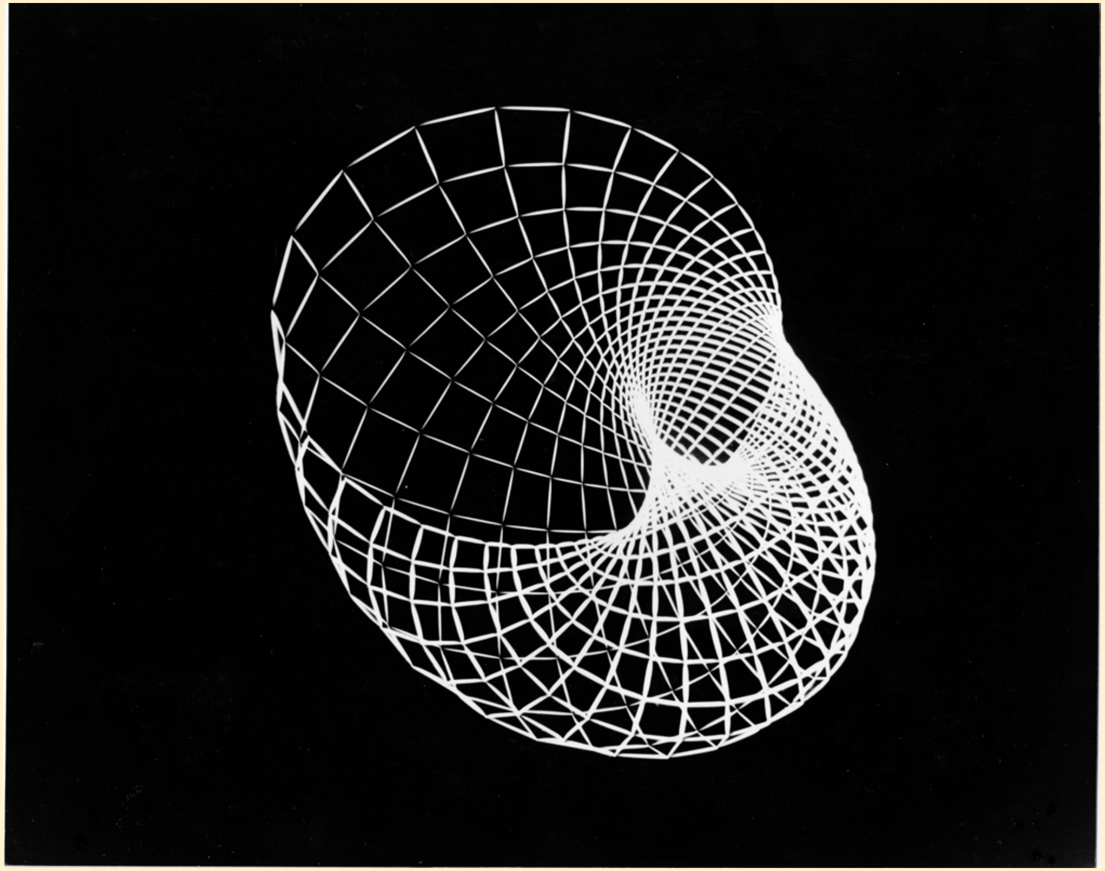
$$e_0 \cdot e_0 = e_1 \cdot e_1 = +1,\ e_2 \cdot e_2 = -1,\ e_3 \cdot e_3 = 0,\ e_i \cdot e_j = 0 \text{ for } i \neq j$$



What is Cayley-Klein geometry?

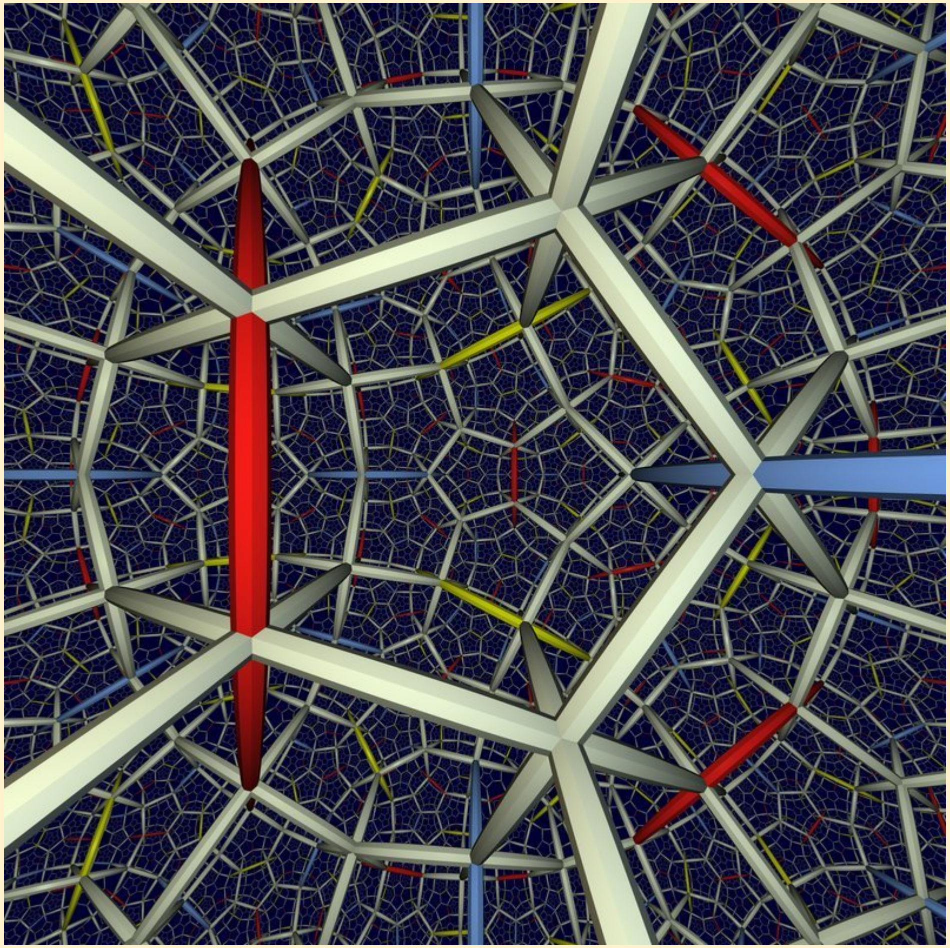
Signature of Q	κ	Space	Symbol
$(n+1,0,0)$	+1	elliptic	$\mathbf{Ell^n}, \mathbf{S^n}$
$(n,1,0)$	-1	hyperbolic	$\mathbf{H^n}$
$"(n,0,1)"$	0	euclidean	$\mathbf{E^n}$

3D Examples



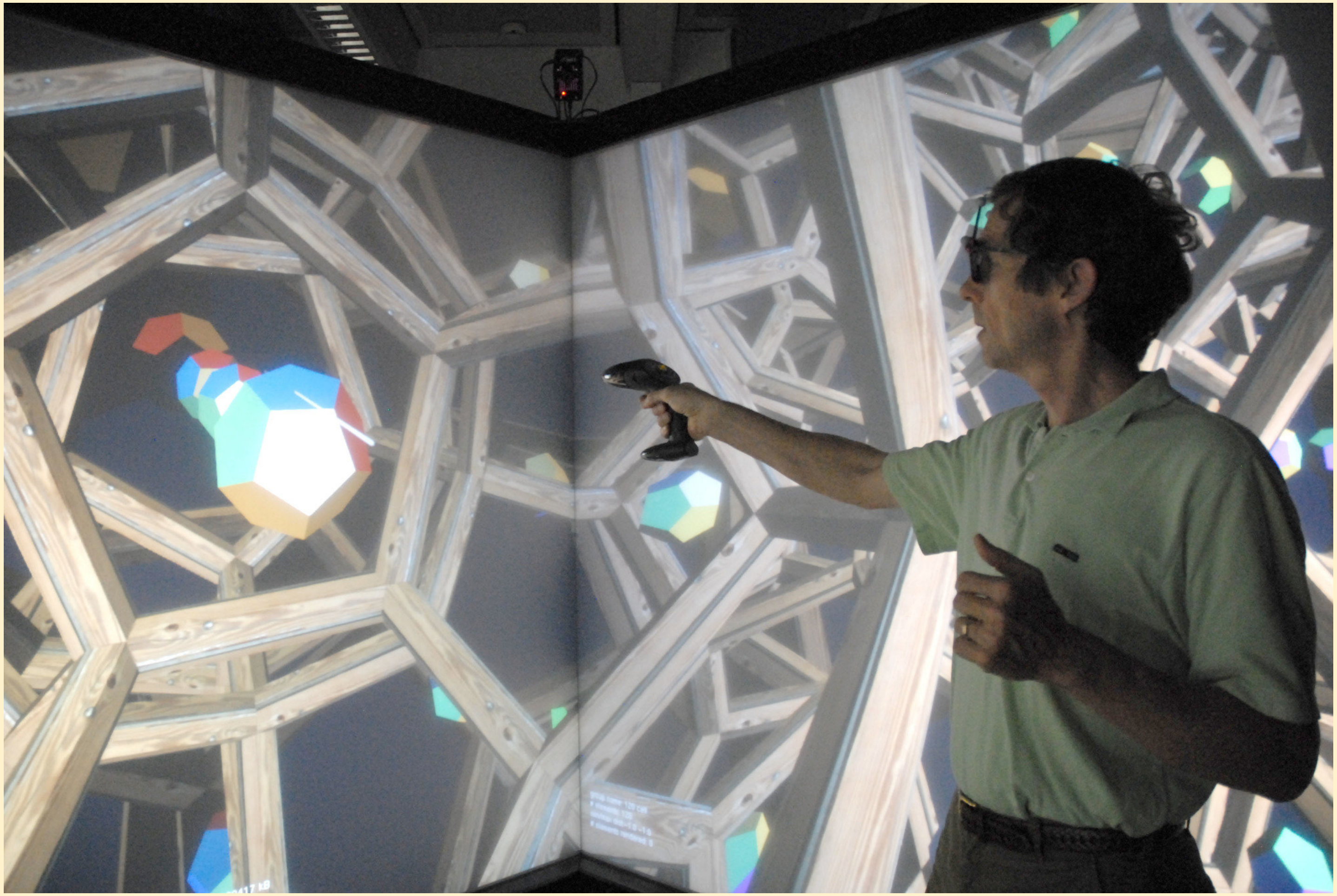
The Sudanese Moebius band in S^3 discovered by Sue Goodman and Dan Asimov, visualized in UNC-CH Graphics Lab, 1979.

3D Examples



Tessellation of H^3 with regular right-angled dodecahedra (from "Not Knot", Geometry Center, 1993).

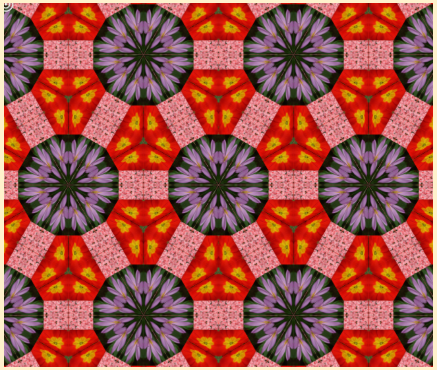
3D Examples

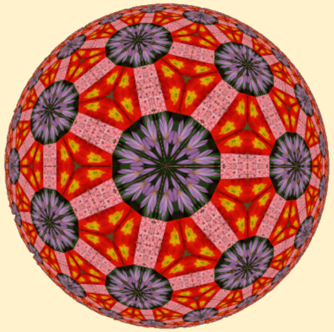


The 120-cell, a tessellation of the 3-sphere $\mathbf{S}^3$

(PORTAL VR, TU-Berlin, 18.09.09)

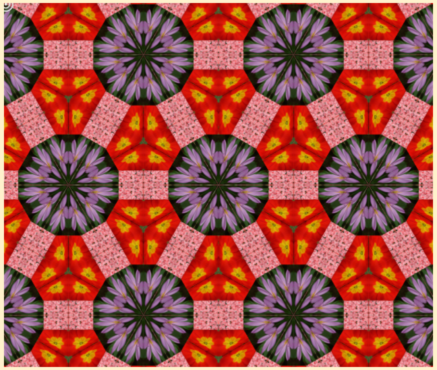
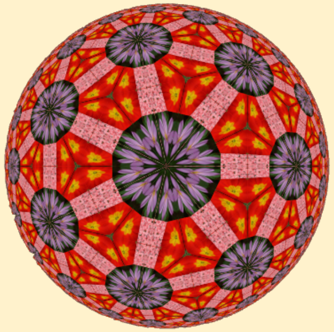
Cayley-Klein geometries for $n = 2$





Name	**elliptic**	**euclidean**	**hyperbolic**
signature	(3,0,0)	"(2,0,1)"	(2,1,0)
null points	$x^2+y^2+z^2=0$		$x^2+y^2-z^2=0$

Cayley-Klein geometries for $n = 2$

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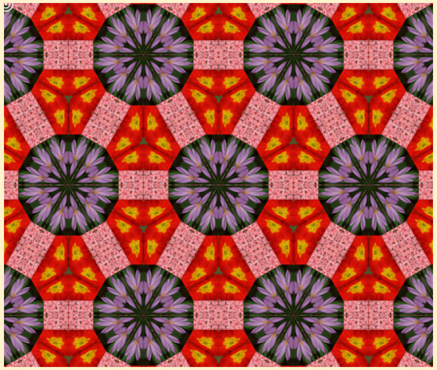
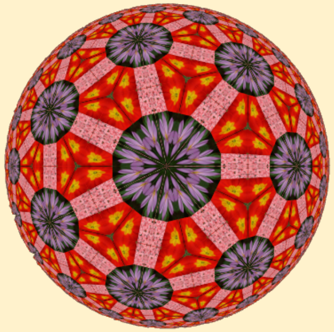
Example Cayley-Klein geometries for $n = 2$

Name	elliptic	euclidean	hyperbolic
signature	(3,0,0)	"(2,0,1)"	(2,1,0)
~~null points~~	$\cancel{x^2+y^2+z^2=0}$	$\cancel{z^2=0}$	$\cancel{x^2+y^2-z^2=0}$
null lines*	$a^2+b^2+c^2=0$	$a^2+b^2=0$	$a^2+b^2-c^2=0$

*The line $ax+by+cz=0$ has line coordinates (a,b,c).

Question

What is the best way
to do Cayley-Klein geometry
on the computer?

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2019

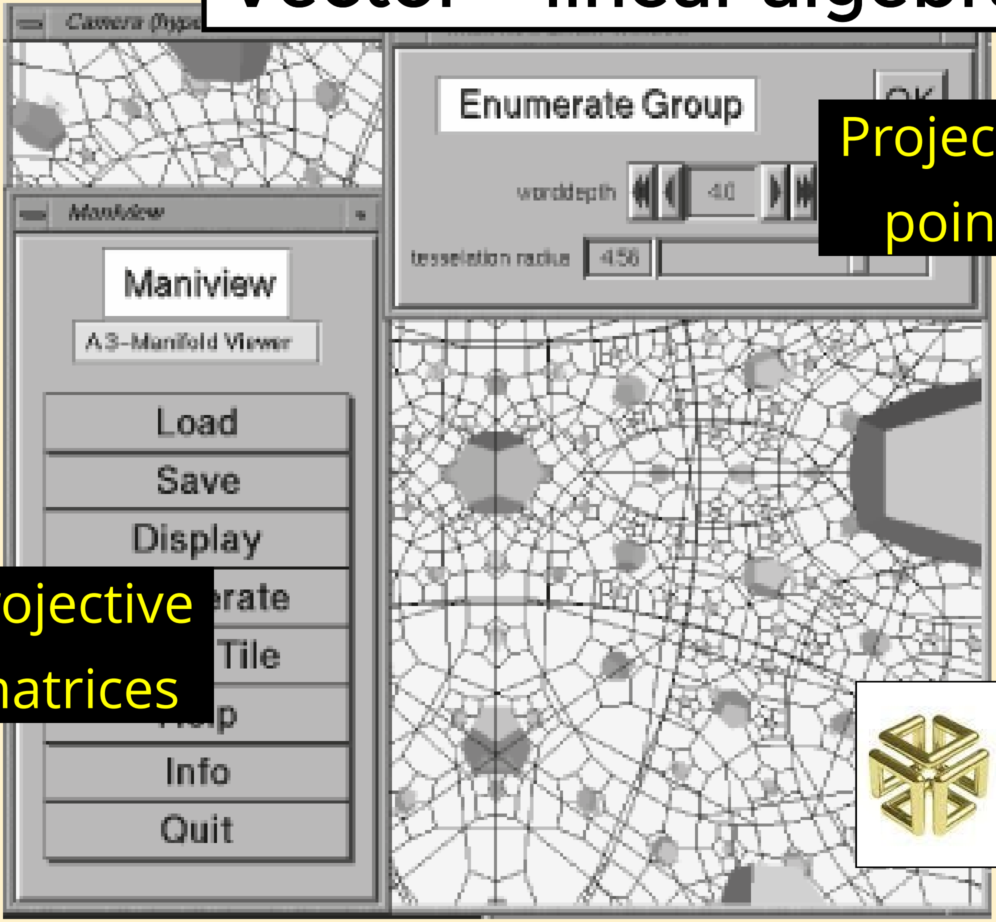
Vector + linear algebra

Vector + linear algebra

Projective points

Projective matrices

Vector + linear algebra

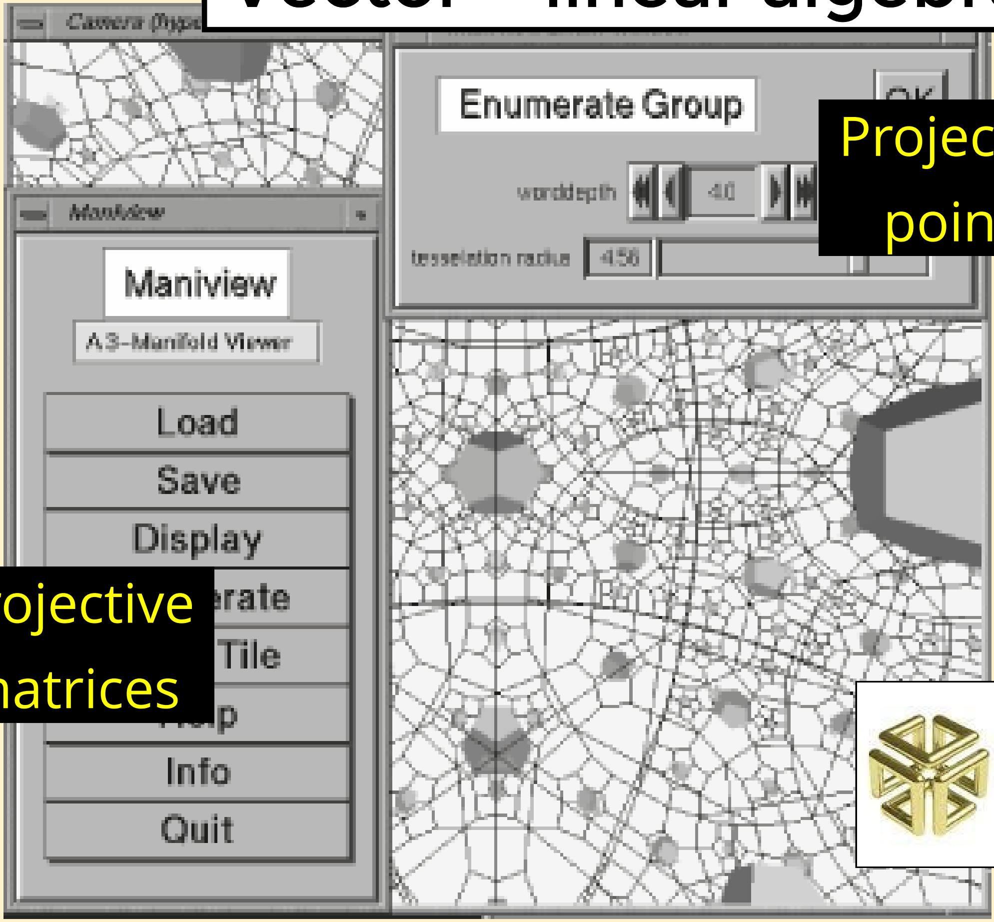


Projective points

Projective matrices

SiliconGraphics

Vector + linear algebra



Projective points

Projective matrices

SiliconGraphics

But it's 2019 now. Can we do better?

Cayley-Klein programmer's wish list

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Coordinate-free

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Uniform rep'n for points, lines, and planes

Coordinate-free

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Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Coordinate-free

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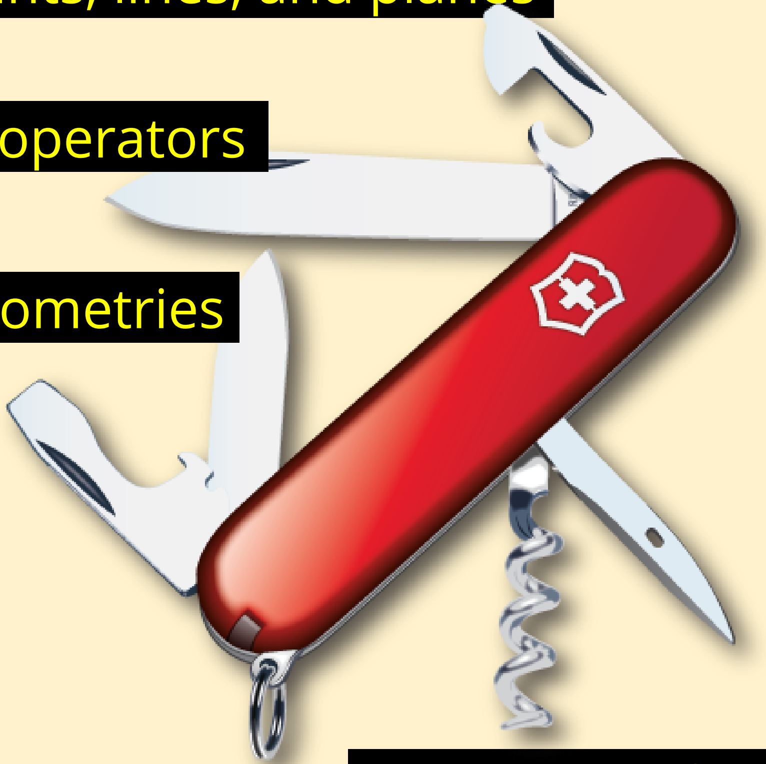
Single, uniform rep'n for isometries



Coordinate-free

Cayley-Klein programmer's wish list

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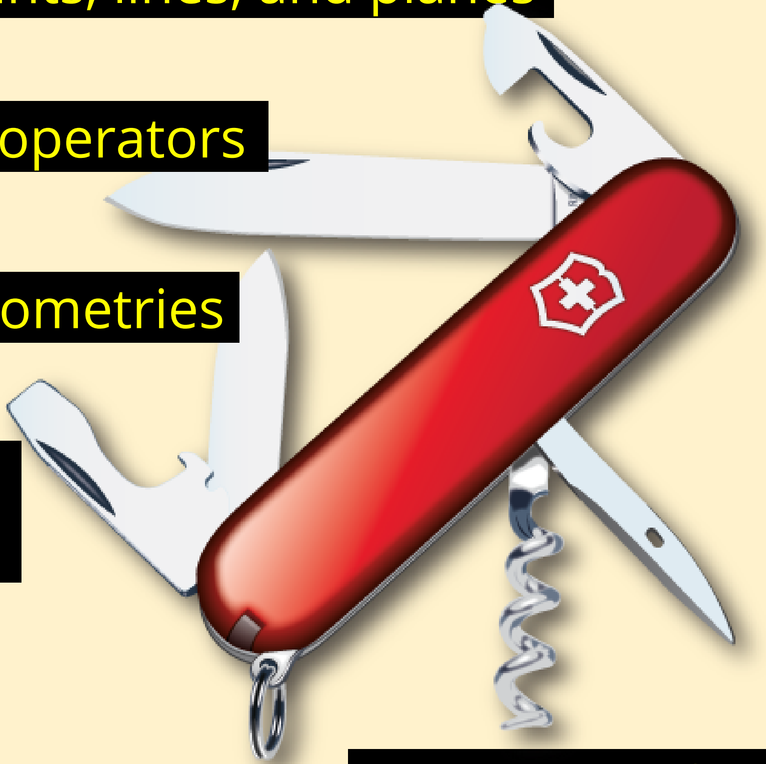
Compact expressions for classical geometric results



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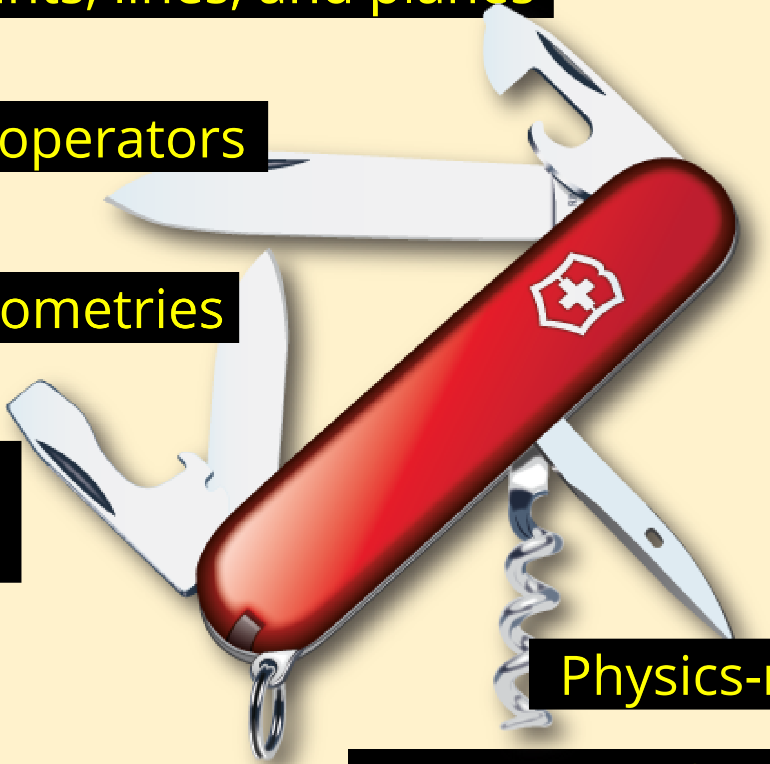

Physics-ready

Coordinate-free

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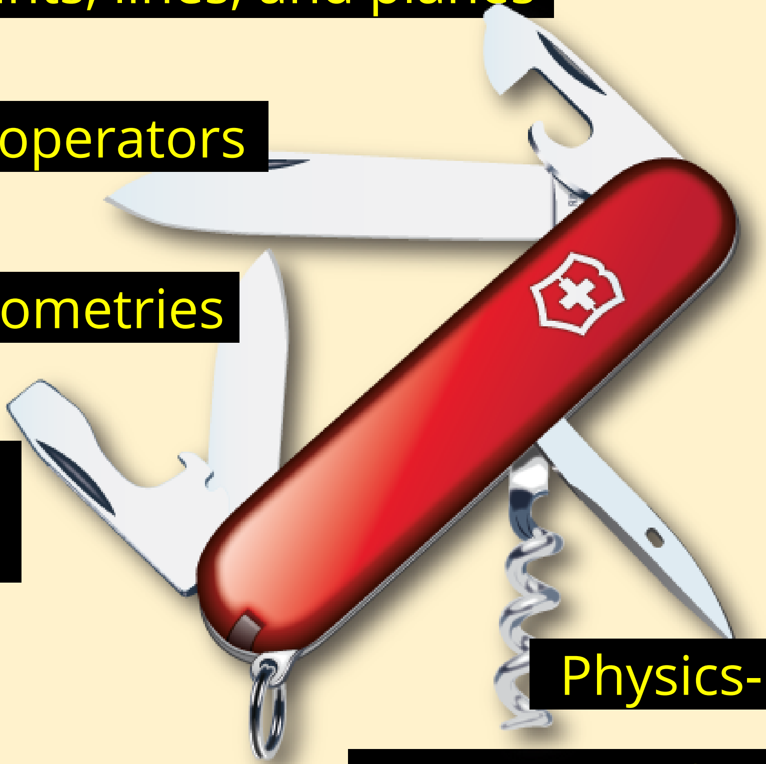


Physics-ready

Coordinate-free

Metric-neutral

Cayley-Klein programmer's wish list

Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Single, uniform rep'n for isometries

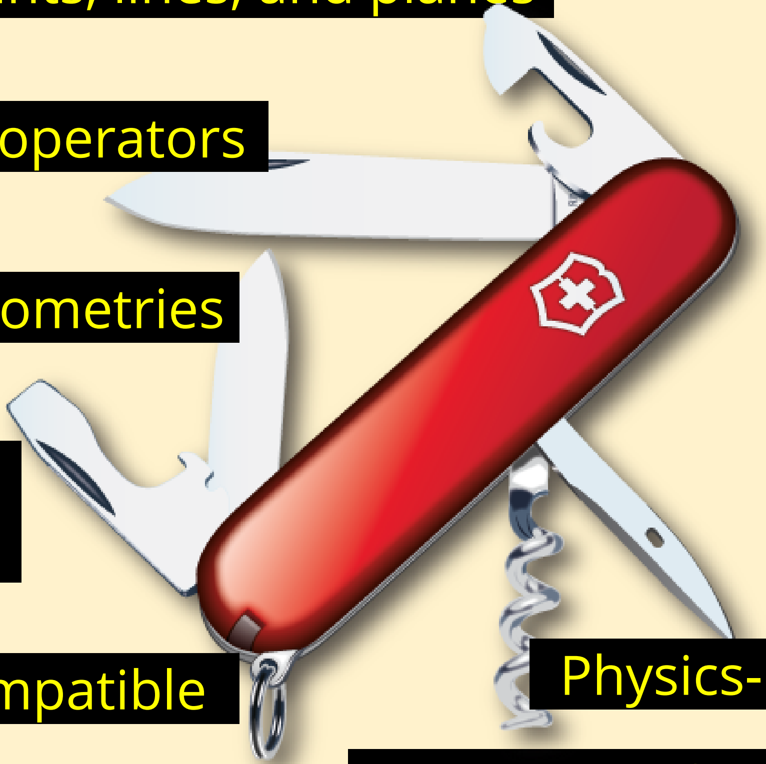
Compact expressions for classical geometric results

Backwards compatible

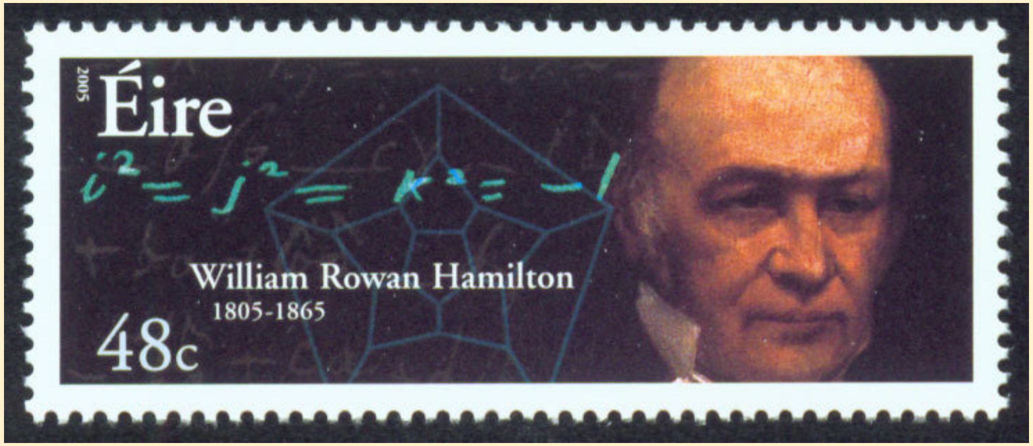
Metric-neutral

Coordinate-free

Physics-ready



Partial solutions: Quaternions (1843)



A 4D algebra generated by units $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying:

$$1^2 = 1, \;\; \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$
$$\mathbf{ij} = -\mathbf{ji}, \ldots$$

Quaternions

Quaternions $\mathbb{H}$	$s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Im. quaternions $\mathbb{IH}$	$\mathbf{v} := x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \;\Leftrightarrow (x, y, z) \in \mathbb{R}^3$
Unit quaternions $\mathbb{U}$	$\{\mathbf{g} \in \mathbb{H} \mid \mathbf{g}\overline{\mathbf{g}} = 1\}$

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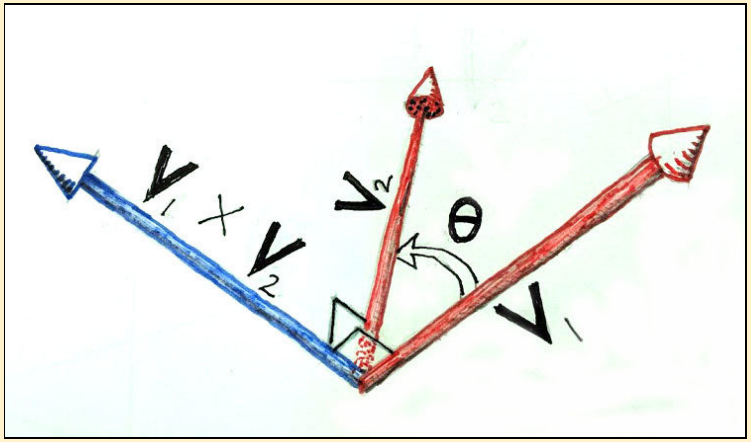
I. Geometric product:

$$\mathbf{v}_1\mathbf{v}_2 = -\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_2$$



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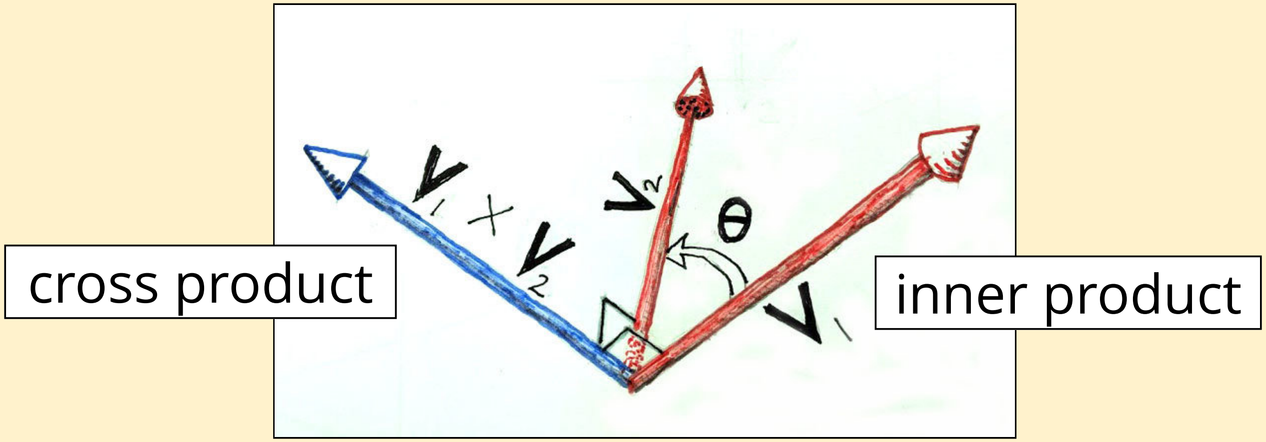

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Unit quaternions $\mathbb{U}$	$\{\mathbf{g} \in \mathbb{H} \mid \mathbf{g}\overline{\mathbf{g}} = 1\}$

II. Rotations via sandwiches:

1. For $\mathbf{g} \in \mathbb{U}$, there exists $\mathbf{x} \in \mathbb{IH}$ so that

$$\mathbf{g} = \cos(t) + \sin(t)\mathbf{x} = e^{t\mathbf{x}}$$

2. For any $\mathbf{v} \in \mathbb{IH}$ $(\cong \mathbb{R}^3)$, the "sandwich"

$$\mathbf{g}\mathbf{v}\overline{\mathbf{g}}$$

rotates $\mathbf{v}$ around the axis $\mathbf{x}$ by an angle $2t$.

3. Comparison to matrices.

Quaternions

Quaternions $\mathbb{H}$	$s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Im. quaternions $\mathbb{IH}$	$\mathbf{v} := x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \ \Leftrightarrow (x, y, z) \in \mathbb{R}^3$
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Advantages

I. Geometric product

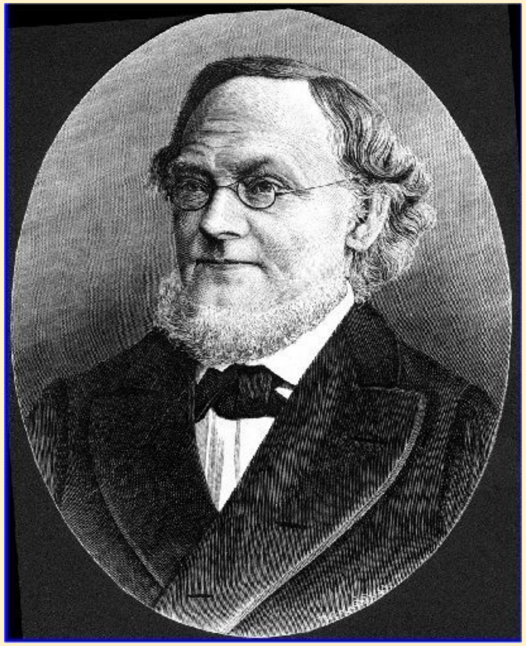
II. Rotations as sandwiches

Disadvantages

I. Only applies to points/vectors

II. Special case $\mathbb{R}^3$

Partial solutions: Grassmann algebra

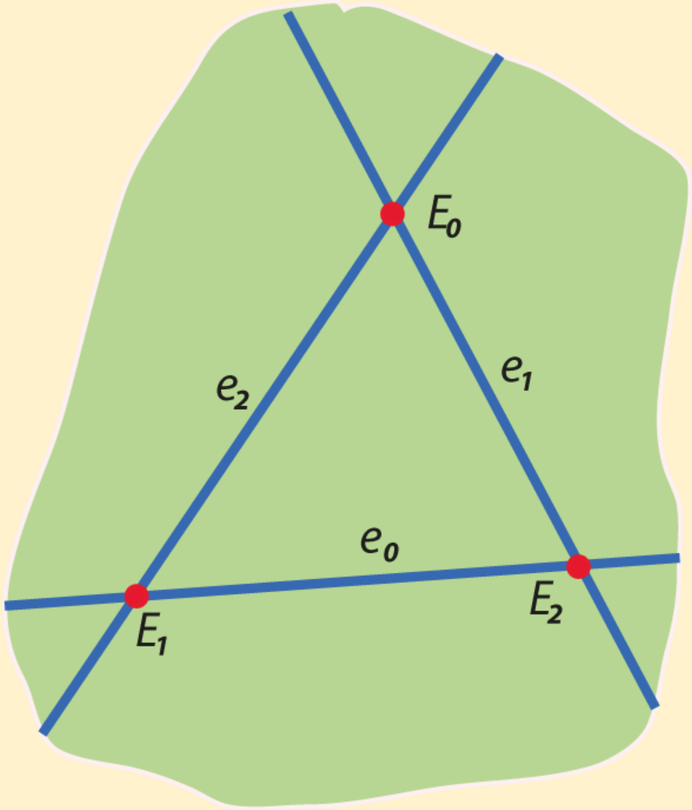


Hermann Grassmann (1809-1877)

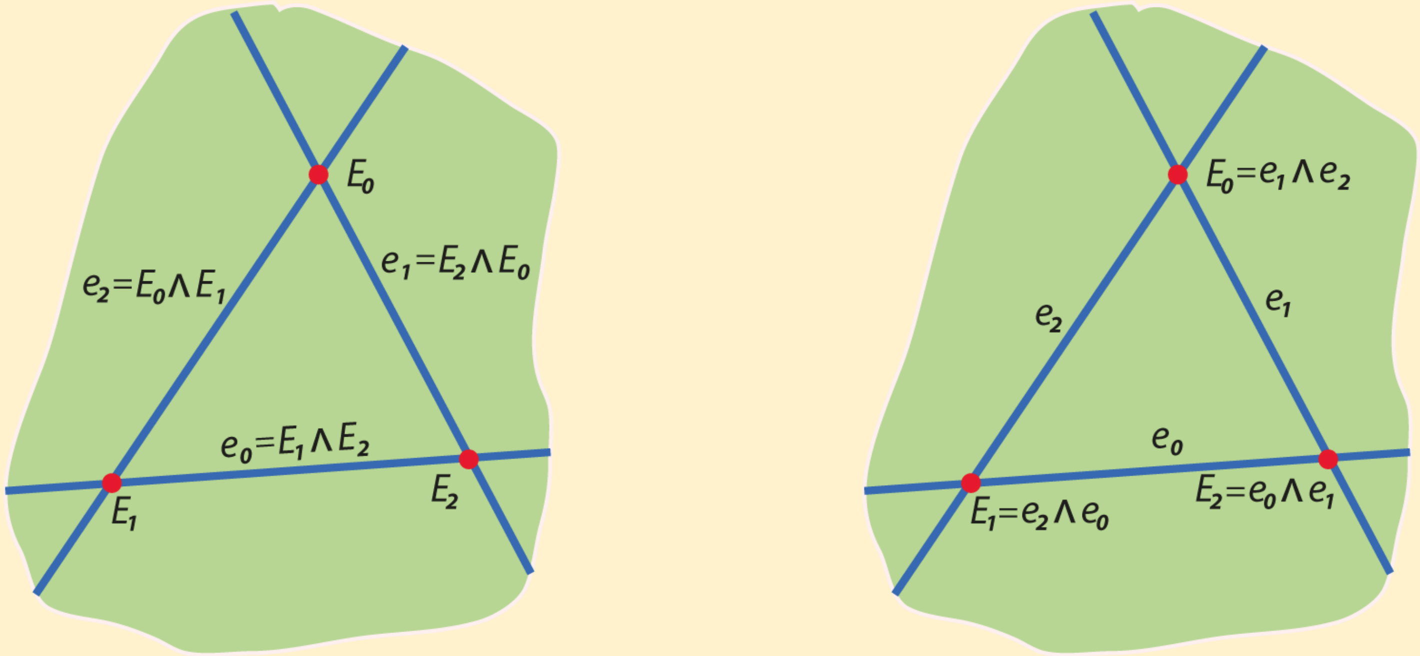
Ausdehnungslehre (1844)

Grassmann algebra

The wedge ($\wedge$) product in $\mathbb{R}P^2$ and $\mathbb{R}P^{2*}$



Grassmann algebra



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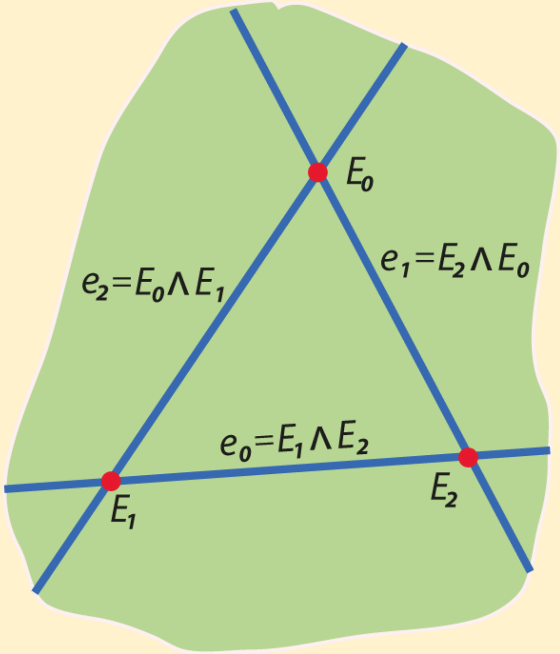
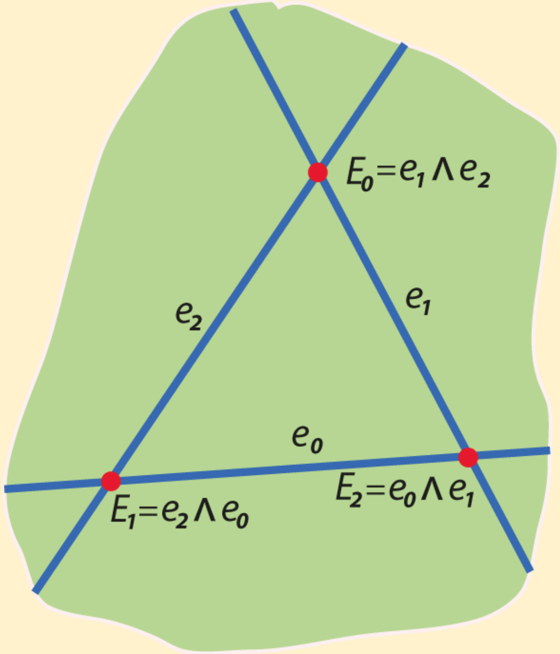


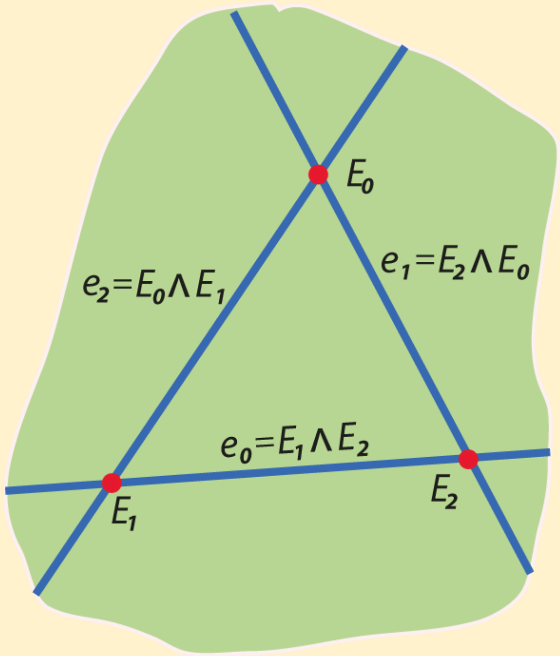


Standard projective

$\mathbf{x} \wedge \mathbf{y}$ is **join**

yields $\bigwedge \mathbb{R}P^2$

Grassmann algebra



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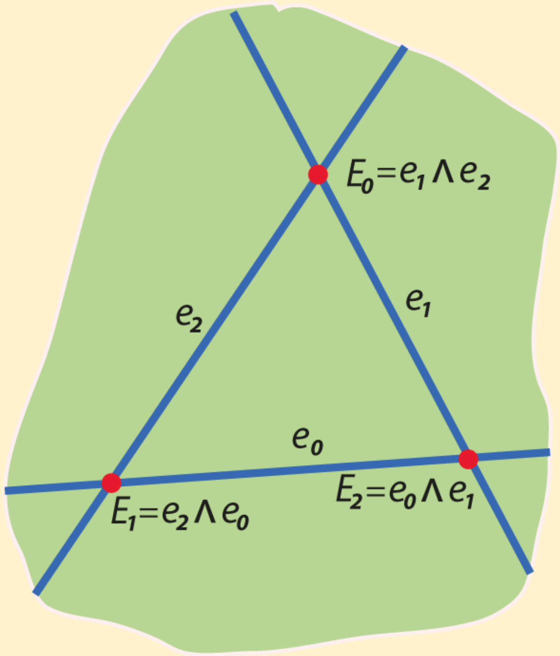


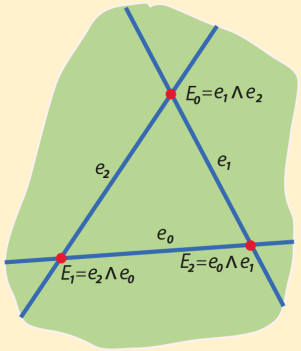
Dual projective

$\mathbf{x}\wedge\mathbf{y}$ is **meet**

yields $\bigwedge \mathbb{R}P^{2*}$

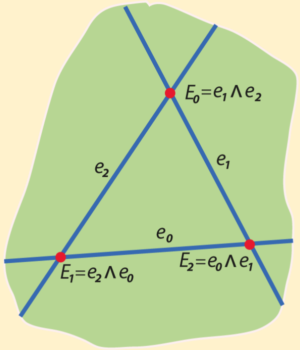
Grassmann algebra

The dual projective
Grassmann algebra $\bigwedge \mathbb{R}P^{2*}$



Grade	Sym	Generators	Dim.	Type
0	$\bigwedge^0$	1	1	Scalar
1	$\bigwedge^1$	$\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$	3	Line
2	$\bigwedge^2$	$\{\mathbf{E}_i = \mathbf{e}_j \wedge \mathbf{e}_k\}$	3	Point
3	$\bigwedge^3$	$\mathbf{I} = \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$	1	Pseudoscalar

Grassmann algebra



The dual projective
Grassmann algebra $\bigwedge \mathbb{R}P^{2*}$

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We will be using $\bigwedge \mathbb{R}P^{n*}$ for the rest of the talk.

Grassmann algebra

The wedge ($\wedge$) product in $\mathbb{R}P^2$

Properties of $\wedge$

1. **Antisymmetric**: For 1-vectors $\mathbf{x}$, $\mathbf{y}$:

$$\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$$
$$\mathbf{x} \wedge \mathbf{x} = 0$$

2. **Subspace lattice**: For linearly independent subspaces $\mathbf{x} \in \bigwedge^k, \mathbf{y} \in \bigwedge^m$, $\mathbf{x} \wedge \mathbf{y} \in \bigwedge^{k+m}$ is the subspace spanned by $\mathbf{x}$ and $\mathbf{y}$ otherwise it's zero.

Note: The regressive (join) product $\vee$ is also available.

(Then it's called a Grassmann-Cayley algebra.)

Grassmann algebra

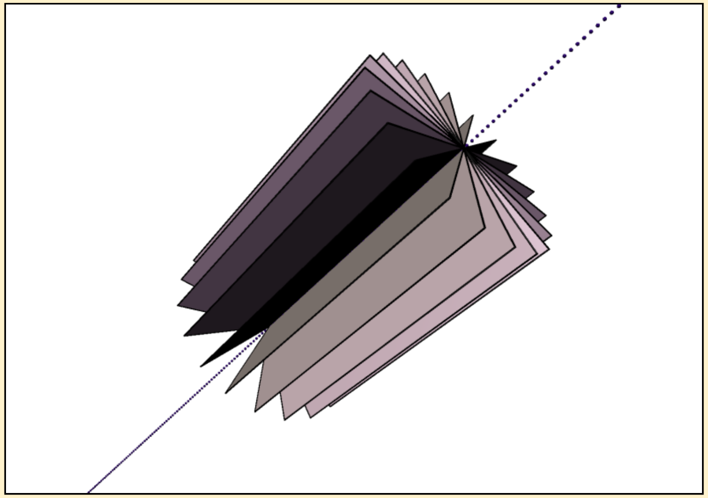
Note: *spanning subspace* means different things in standard and dual setting. In 3D:

Standard: a line is the subspace spanned by two points.

Dual: a line is the subspace spanned by two planes.

Point range

Spear



Plane pencil

Axis

Grassmann algebra

Advantages

1. Points, lines, and planes are equal citizens.
2. "Parallel-safe" meet and join operators since projective.

Disadvantages

1. Only incidence (projective), no metric.

Clifford's geometric algebra

William Kingdon Clifford (1845-1879)

"*Applications of Grassmann's extensive algebra*" (1878):

His stated aim: to combine quaternions with Grassmann algebra.

Clifford's geometric algebra

Geometric product extends the wedge product and is defined for two 1-vectors as:

$$\mathbf{x}\mathbf{y} := \underbrace{\mathbf{x}\cdot\mathbf{y}}_{\text{0-vector}} + \underbrace{\mathbf{x}\wedge\mathbf{y}}_{\text{2-vector}}$$

where $\cdot$ is the inner product induced by $\mathbf{Q}$.

Since the two terms measure different aspects, the sum is (usually) non-zero.

This product can be extended to the whole Grassmann algebra to produce the **geometric algebra** $\mathbf{P}(\mathbb{R}^*_{p,n,z})$.

Clifford's geometric algebra

Geometric product extends the wedge product and is defined for two 1-vectors as:

$$\mathbf{x}\mathbf{y} := \underbrace{\mathbf{x}\cdot\mathbf{y}}_{\text{0-vector}} + \underbrace{\mathbf{x}\wedge\mathbf{y}}_{\text{2-vector}}$$

measures sameness

measures difference

where $\cdot$ is the inner product induced by $\mathbf{Q}$.

Since the two terms measure different aspects,
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This product can be extended to the whole Grassmann algebra to produce the **geometric algebra** $\mathbf{P}(\mathbb{R}^*_{p,n,z})$.

Projective geometric algebra

We call an algebra constructed in this way a *projective geometric algebra* (PGA).

We are interested in $\mathbf{P}(\mathbb{R}^*_{3,0,0})$, $\mathbf{P}(\mathbb{R}^*_{2,1,0})$, $\mathbf{P}(\mathbb{R}^*_{2,0,1})$.

We sometimes write $\mathbf{P}(\mathbb{R}^*_\kappa)$ and leave the metric open.

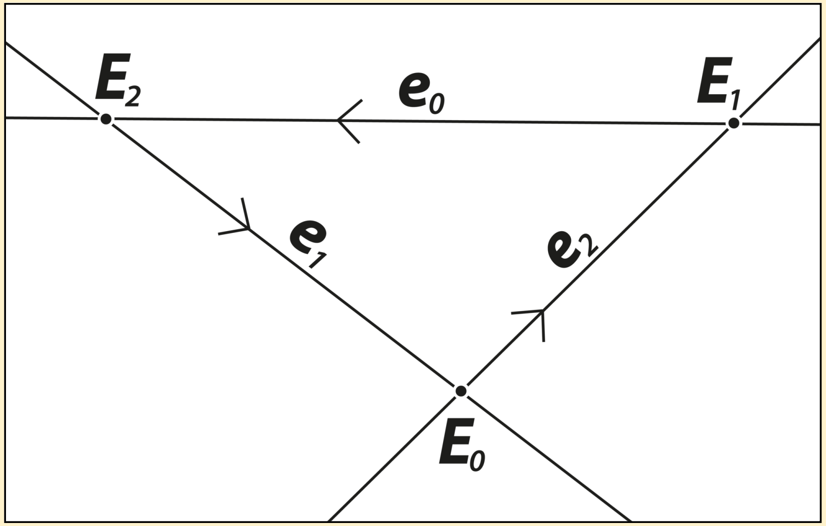
We can choose a fundamental triangle so that:

$$\mathbf{e}_0^2 = \kappa, \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$$

$$(\kappa \in \{-1, 0, 1\})$$

$$\mathbf{E}_k := \mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$$

$$\mathbf{I} := \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$$



2D PGA

Example: Two lines, let $e_0^2 = \kappa$

$$\mathbf{a} = a_0 e_0 + a_1 e_1 + a_2 e_2$$

$$\mathbf{b} = b_0 e_0 + b_1 e_1 + b_2 e_2$$

$$\mathbf{ab} = (a_0 b_0 e_0^2 + a_1 b_1 e_1^2 + a_2 b_2 e_2^2)$$

$$+(a_0 b_1 - a_1 b_0) e_0 e_1 + (a_1 b_2 - a_2 b_1) e_1 e_2 + (a_0 b_2 - a_2 b_0) e_0 e_2$$

$$= (a_0 b_0 \kappa + a_1 b_1 + a_2 b_2)$$

$$+(a_1 b_2 - a_2 b_1) E_0 + (a_2 b_0 - a_0 b_2) E_1 + (a_0 b_1 - a_1 b_0) E_2$$

$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

2D PGA

Example: Two lines, let $e_0^2 = \kappa$

$$
\begin{aligned}
\mathbf{a} &= a_0e_0 + a_1e_1 + a_2e_2 \\
\mathbf{b} &= b_0e_0 + b_1e_1 + b_2e_2 \\
\mathbf{ab} &= (a_0b_0e_0^2 + a_1b_1e_1^2 + a_2b_2e_2^2) \\
&+(a_0b_1 - a_1b_0)e_0e_1 + (a_1b_2 - a_2b_1)e_1e_2 + (a_0b_2 - a_2b_0)e_0e_2 \\
&= (a_0b_0\kappa + a_1b_1 + a_2b_2) \\
&+\boxed{(a_1b_2 - a_2b_1)E_0 + (a_2b_0 - a_0b_2)E_1 + (a_0b_1 - a_1b_0)E_2} \\
&= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}
\end{aligned}
$$

Looks like cross product but is the point incident to both lines.

2D PGA

Example: $\kappa = 1$ and $c = \frac{1}{\sqrt{2}}$ and

$$\mathbf{a} = e_0, \quad \mathbf{b} = ce_0 + ce_1$$

Then $\mathbf{a}^2 = \mathbf{b}^2 = 1$.

$\mathbf{a}$ is the equator great circle $z = 0$ and $\mathbf{b}$ is tilted up from it an angle of 45°.

$$\mathbf{ab} = c + E_2$$

Check: $\cos^{-1}(c) = 45^\circ$ and $\mathbf{E}_2$ is the common point.

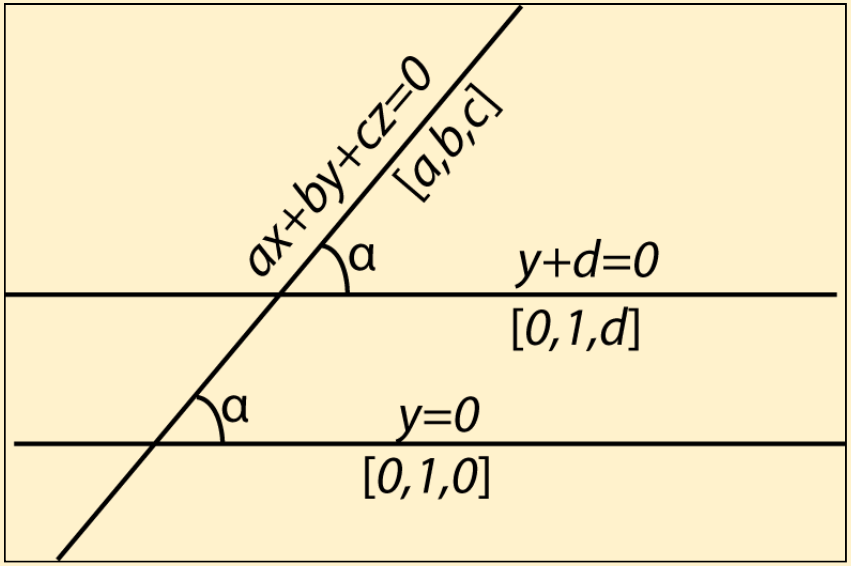
2D Euclidean PGA

We can now explain why $P(\mathbb{R}^*_{2,0,1})$ is the right choice for the euclidean plane.

The inner product of two lines is

$$\mathbf{a} \cdot \mathbf{b} = (a_0 b_0 \kappa + a_1 b_1 + a_2 b_2)$$

For a euclidean line changing a_0 or b_0 doesn't change the direction of the line. It just moves it parallel to itself. This means $e_0^2 = 0$.



Clifford's geometric algebra

	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{e}_0$	$\mathbf{e}_0$	κ	$\mathbf{E}_2$	$-\mathbf{E}_1$	$\mathbf{I}$	$-\kappa\mathbf{e}_2$	$\kappa\mathbf{e}_1$	$\kappa\mathbf{E}_0$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{E}_2$	$\mathbf{1}$	$\mathbf{E}_0$	$\mathbf{e}_2$	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_1$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{E}_1$	$-\mathbf{E}_0$	$\mathbf{1}$	$-\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{I}$	$\mathbf{E}_2$
$\mathbf{E}_0$	$\mathbf{E}_0$	$\mathbf{I}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{1}$	$-\mathbf{E}_2$	$\mathbf{E}_1$	$-\mathbf{e}_0$
$\mathbf{E}_1$	$\mathbf{E}_1$	$\kappa\mathbf{e}_2$	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_2$	$-\kappa$	$-\kappa\mathbf{E}_0$	$-\kappa\mathbf{e}_1$
$\mathbf{E}_2$	$\mathbf{E}_2$	$-\kappa\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{I}$	$-\mathbf{E}_1$	$\kappa\mathbf{E}_0$	$-\kappa$	$-\kappa\mathbf{e}_2$
$\mathbf{I}$	$\mathbf{I}$	$\kappa\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$-\mathbf{e}_0$	$-\kappa\mathbf{e}_1$	$-\kappa\mathbf{e}_2$	$-\kappa$

Multiplication table for 2D PGA. $\kappa \in \{-1, 0, 1\}$

2D PGA Preliminaries

1. We can **normalize** a proper line $\mathbf{m}$ or point $\mathbf{P}$ so that:

$$\mathbf{m}^2 = 1, \quad \mathbf{P}^2 = -\boldsymbol{\kappa}$$

1a. Elements such that $\mathbf{x}^2 = 0$ are called *ideal*.

1b. Formulas given below often assume normalized arguments.

PGA: 2-way products

2. **Multiplication with $\mathbf{I}$:** For any k-vector $\mathbf{x}$, $\mathbf{x}^{\perp} := \mathbf{x}\mathbf{I}$ is the *orthogonal complement* of $\mathbf{x}$.

Example: $\mathbf{e}_0\mathbf{I} = \kappa\mathbf{e}_1\mathbf{e}_2$. The only thing left in $\mathbf{I}$ is what **isn't** in $\mathbf{X}$.

2a. In the euclidean case, $\mathbf{I}^2 = 0$.

PGA: 2-way products

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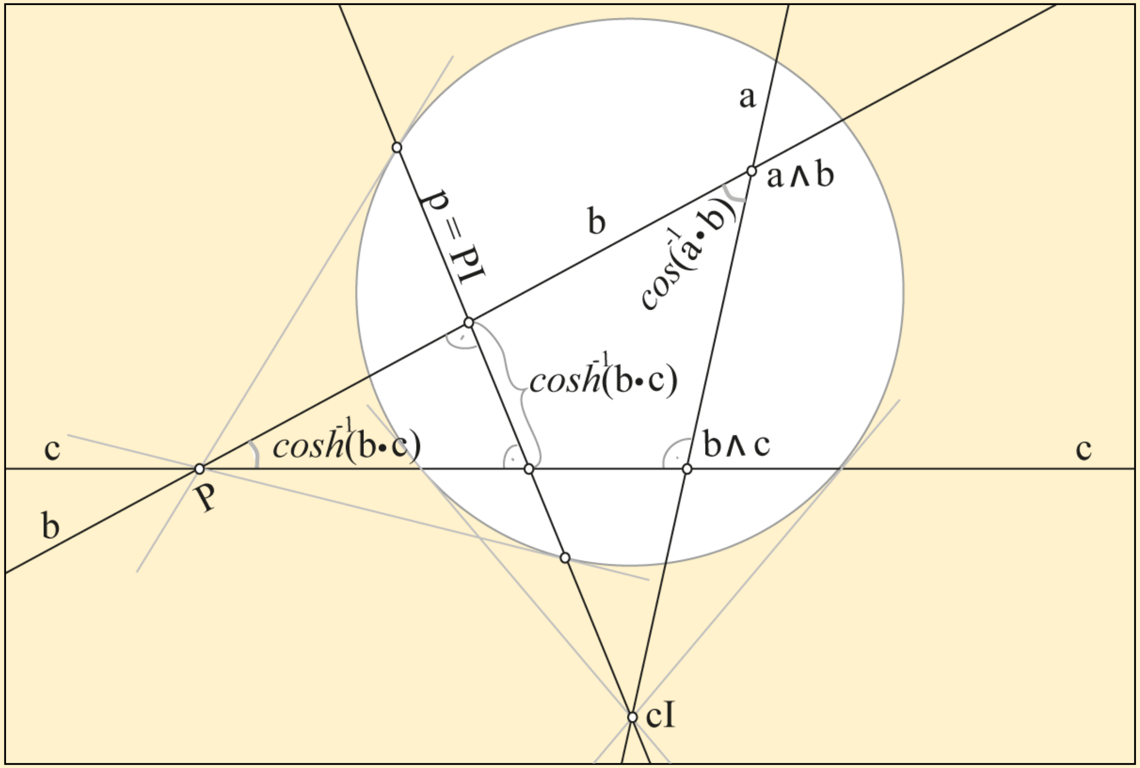
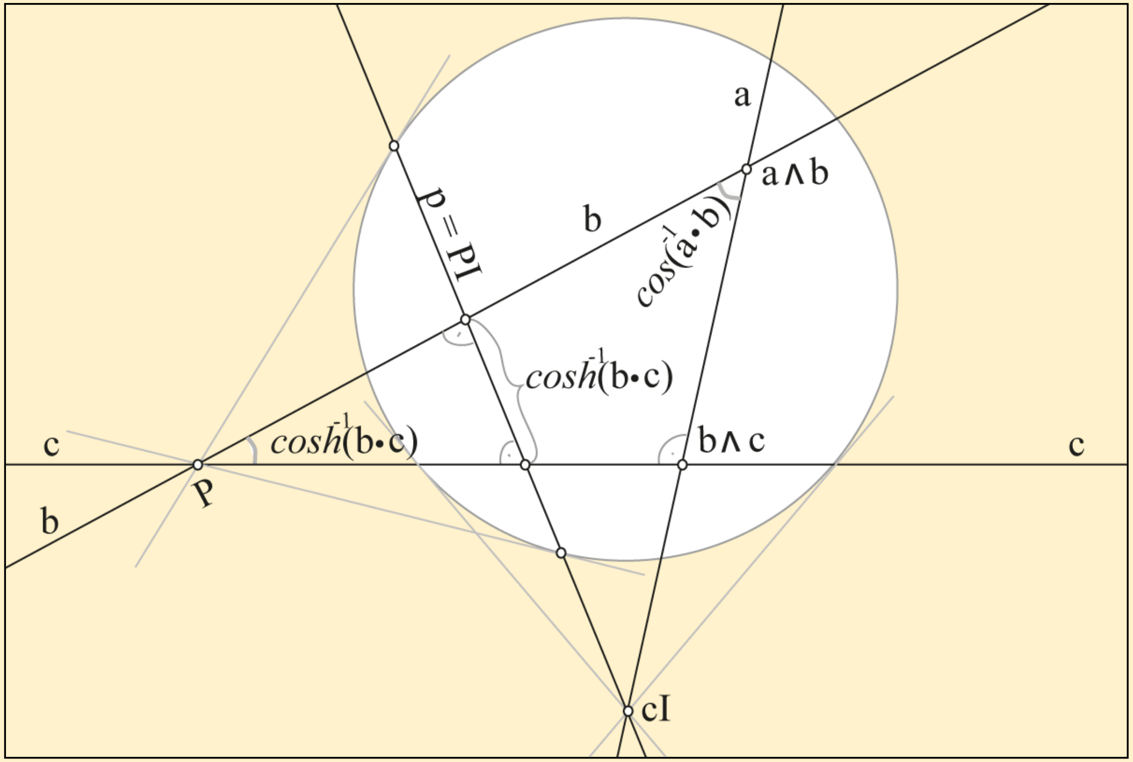


PGA: 2-way products

3. **Product of two proper lines $\mathbf{a}, \mathbf{b}$** that meet at a proper point $\mathbf{P}$:

$$\mathbf{ab} = \cos(t) + \sin(t)\mathbf{P}$$

where t is the angle between the lines (arbitrary κ).

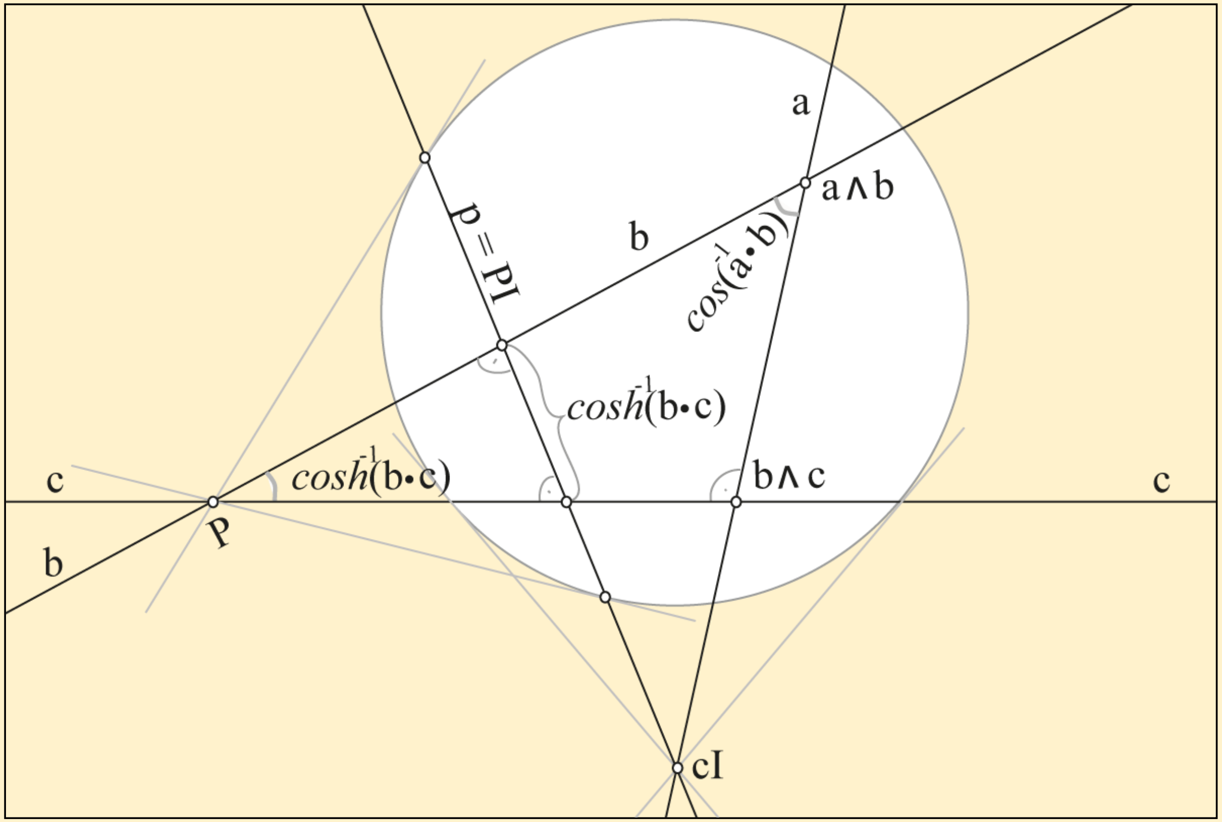


PGA: 2-way products

3a. **Product of two proper lines** $\mathbf{a}, \mathbf{b}$ $\kappa = -1$, $\mathbf{P}$ is hyper-ideal point. Then

$$\mathbf{ab} = cosh(d) + \sinh(d)\mathbf{P}$$

where d is the hyperbolic distance between the lines.

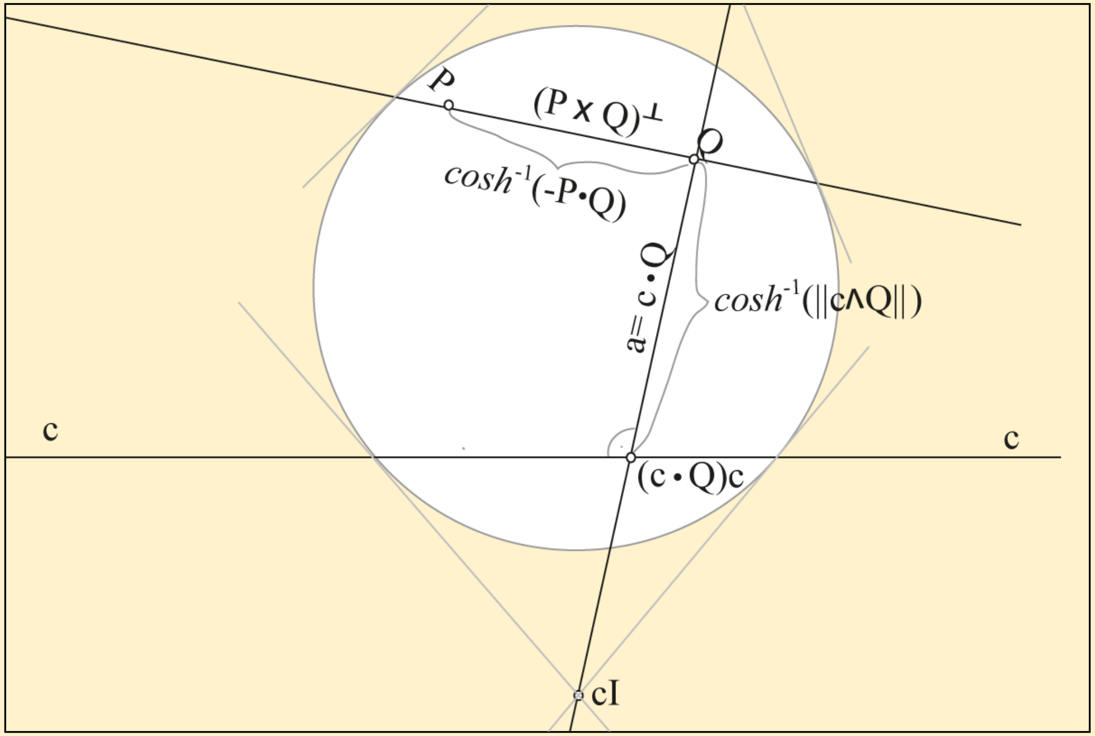


PGA: 2-way products

4. **Product of proper line $\mathbf{c}$ and proper point $\mathbf{Q}$:**

$$\mathbf{cQ} = \mathbf{c}\cdot\mathbf{Q} + (\cosh d)\mathbf{I}\ (= \langle cQ\rangle_1 + \langle cQ\rangle_3)$$

The first term is the line through $\mathbf{Q}$ perpendicular to $\mathbf{c}$, sometimes written $\mathbf{a}^{\perp}_{\mathbf{Q}}$. d is the distance from point to line.

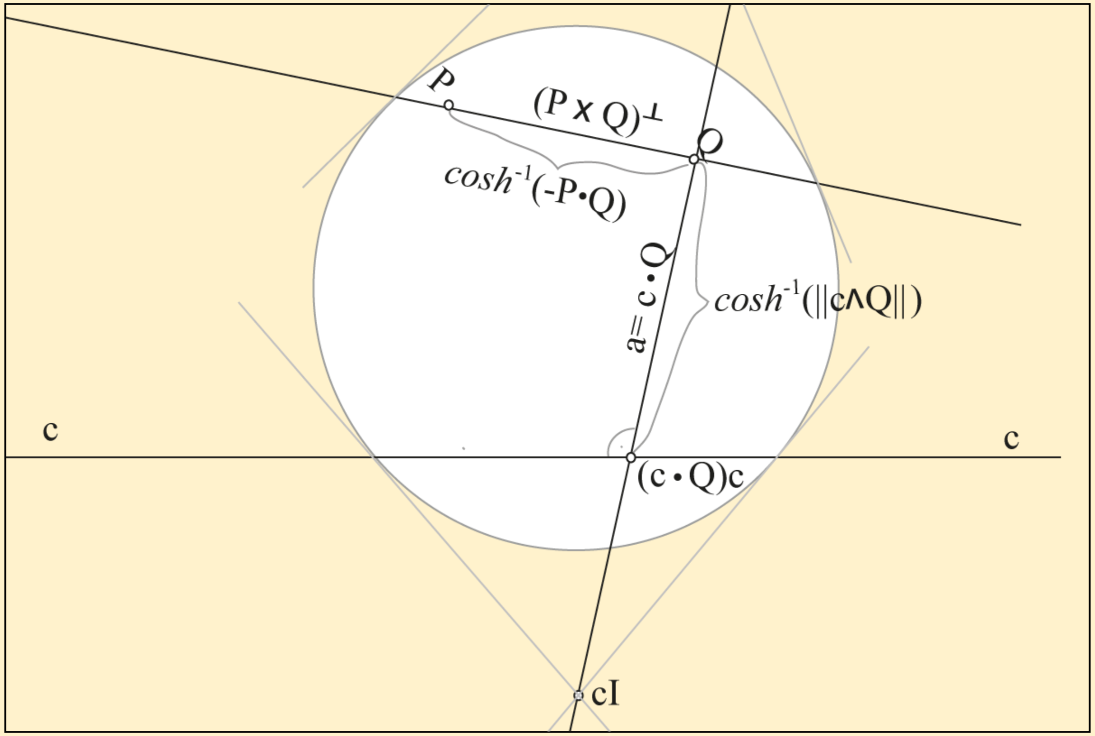


PGA: 2-way products

5. **Product of two proper points $\mathbf{P}, \mathbf{Q}$**. Then

$$\mathbf{PQ} = \cosh(d) + \sinh(d)\mathbf{R}$$

d is the distance between the two points and $\mathbf{R}$ is the normalized form of $\langle \mathbf{PQ} \rangle_2$, which is the common orthogonal $(\mathbf{P} \vee \mathbf{Q})^{\perp}$.

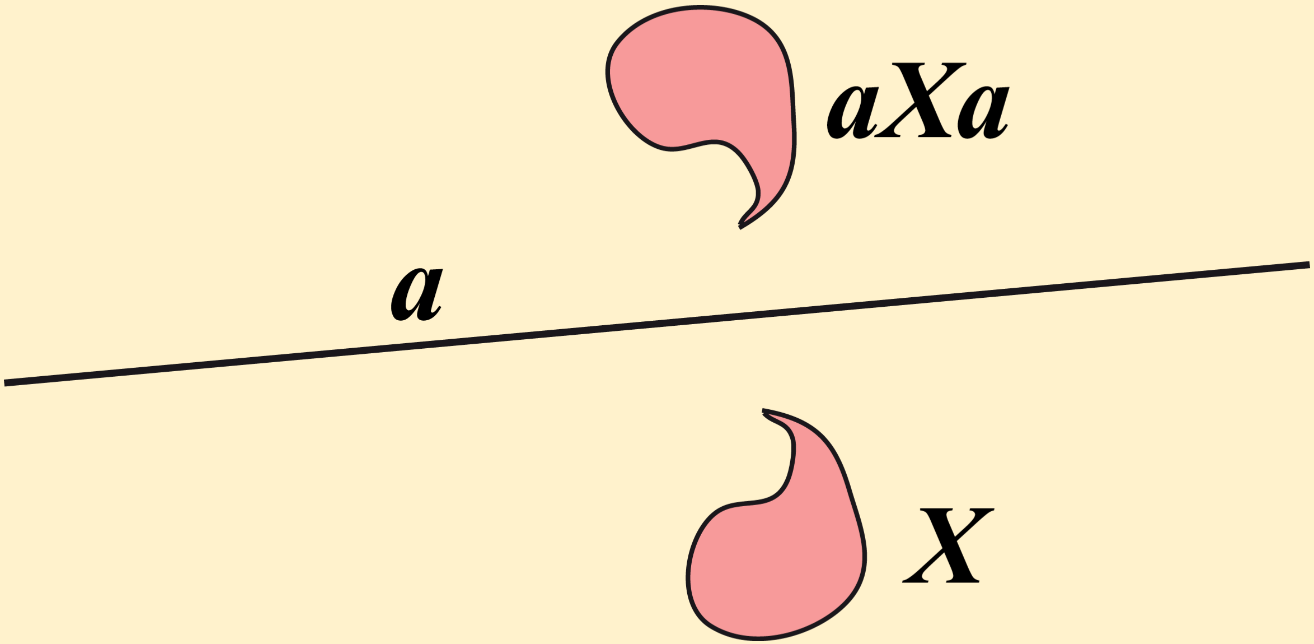


Isometries via 3-way products

Reflections

Consider the 3-way product $\mathbf{X}' = \mathbf{aXa}$, where $\mathbf{a}$ is a proper line and $\mathbf{X}$ is anything.

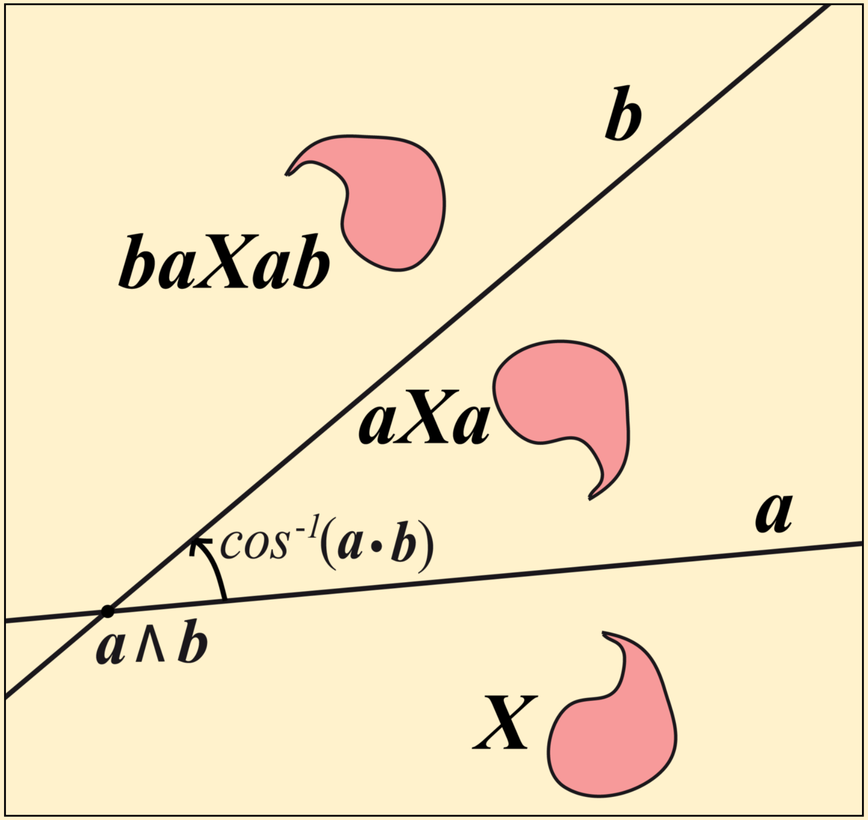
Then $\mathbf{X}'$ is the reflection of $\mathbf{X}$ in the line $\mathbf{a}$.



Isometries via 3-way products

Rotations

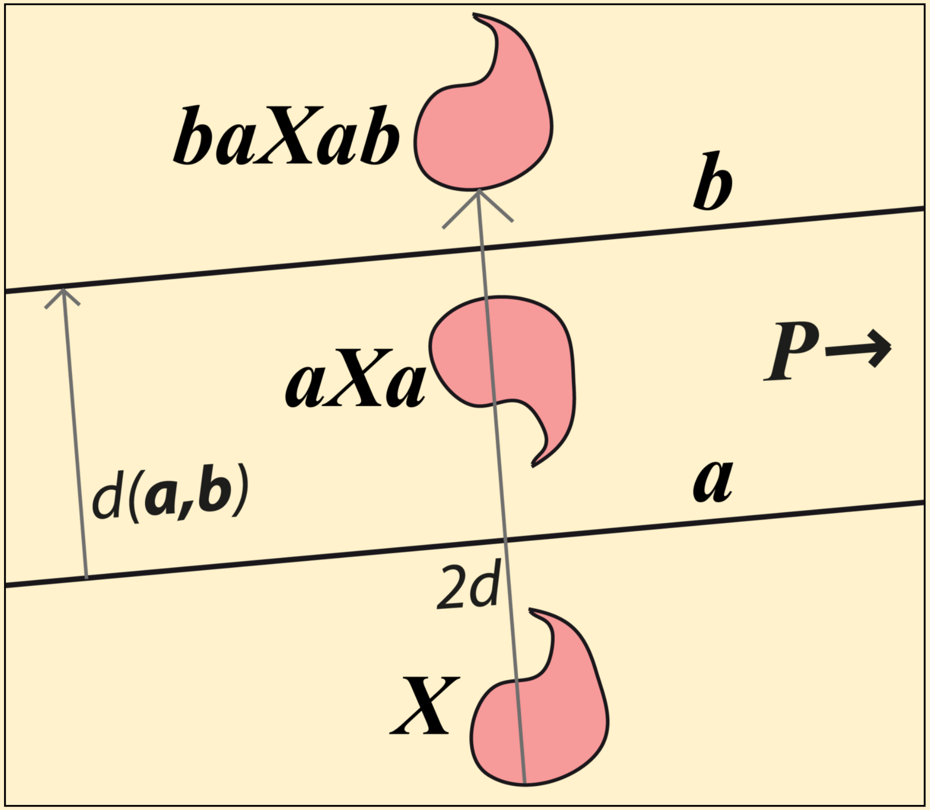
A reflection in a second proper line $\mathbf{b}$ gives $\mathbf{X}' = \mathbf{b}(\mathbf{aXa})\mathbf{b} = (\mathbf{ba})\mathbf{X}(\mathbf{ab})$, by associativity. $\mathbf{r} := \mathbf{ba}$ is called a *rotor* and $X' = RX\widetilde{R}$ where $\widetilde{R}$ is reversal operator.



Isometries via 3-way products

Euclidean translations

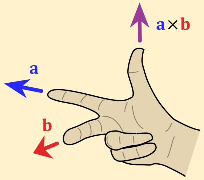
If $\kappa = 0$ and $\mathbf{P}$ is ideal, $\mathbf{X}'$ is a translation of distance 2d, where d is the distance betwen a and b. Similar results for $\kappa = -1$.



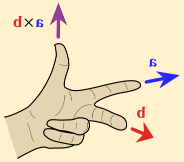
Quaternions in 2D elliptic PGA

	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{e}_0$	$\mathbf{e}_0$	κ	$\mathbf{E}_2$	$-\mathbf{E}_1$	$\mathbf{I}$	$-\kappa\mathbf{e}_2$	$\kappa\mathbf{e}_1$	$\kappa\mathbf{E}_0$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{E}_2$	$\mathbf{1}$	$\mathbf{E}_0$	$\mathbf{e}_2$	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_1$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{E}_1$	$-\mathbf{E}_0$	$\mathbf{1}$	$-\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{I}$	$\mathbf{E}_2$
$\mathbf{E}_0$	$\mathbf{E}_0$	$\mathbf{I}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{1}$	$-\mathbf{E}_2$	$\mathbf{E}_1$	$-\mathbf{e}_0$
$\mathbf{E}_1$	$\mathbf{E}_1$	$\kappa\mathbf{e}_2$	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_2$	$-\kappa$	$-\kappa\mathbf{E}_0$	$-\kappa\mathbf{e}_1$
$\mathbf{E}_2$	$\mathbf{E}_2$	$-\kappa\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{I}$	$-\mathbf{E}_1$	$\kappa\mathbf{E}_0$	$-\kappa$	$-\kappa\mathbf{e}_2$
$\mathbf{I}$	$\mathbf{I}$	$\kappa\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$-\mathbf{e}_0$	$-\kappa\mathbf{e}_1$	$-\kappa\mathbf{e}_2$	$-\kappa$

For $\kappa = 1$, the even sub-algebra (shown in red) is isomorphic to $\mathbb{H}$ under the map



$$\{1, \mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2\} \Leftrightarrow \{1, \mathbf{k}, \mathbf{j}, \mathbf{i}\}$$



Exponentiating bivectors

Every rotor can be produced directly by exponentation of a bivector. When $\mathbf{P}^2 = -1$ then

$$\mathbf{r} := \exp(t\mathbf{P}) = \cos(t) + \sin(t)\mathbf{P}$$

$\mathbf{r}\mathbf{X}\tilde{\mathbf{r}}$ produces a rotation through angle $2t$ around $\mathbf{P}$.

Analogous results hold for $\mathbf{P}^2 = 0 \text{ or } 1$ yielding parabolic or hyperbolic isometries.

The ideal norm

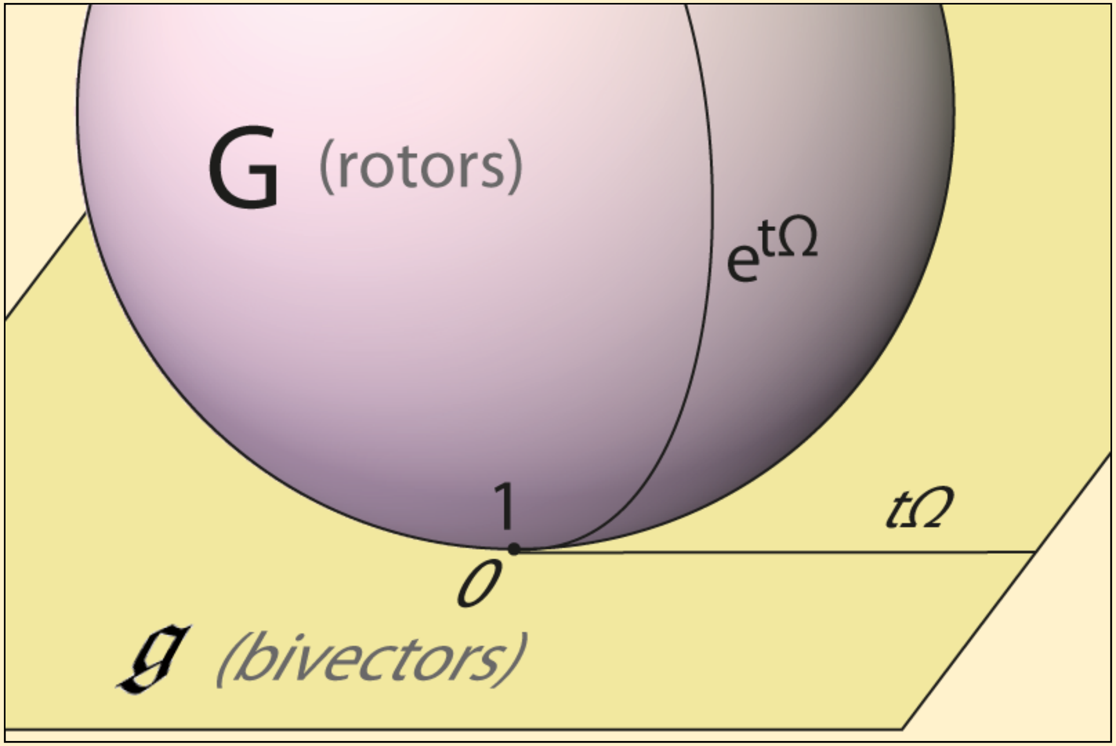
$\mathbf{P}^2 = 0$: how to normalize $\mathbf{P}$ so e^{dP} is a translation of 2d? Time permitting ...

Lie algebra and Lie group

The bivectors $\bigwedge^2$ form the **Lie algebra**.

Define $\mathbf{G}$ to be the elements of the even sub-algebra of norm 1. Then $\mathbf{G}$ is the **Lie group**.

And $\exp : \bigwedge^2 \to \mathbf{G}$ is a 1:1 map up to multiples of 2π (for $n = 3$).

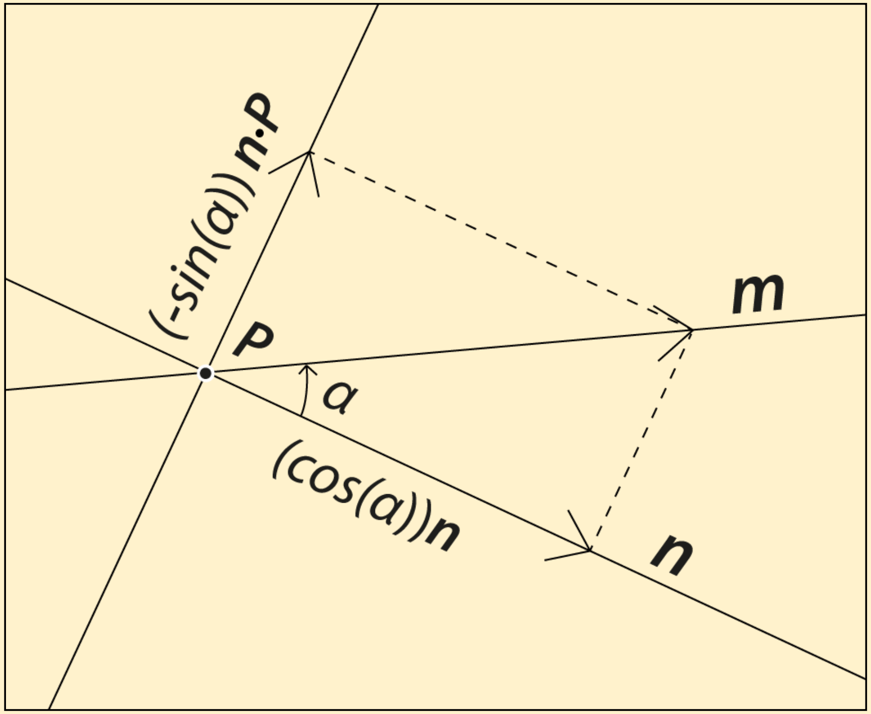


Formula factories via 3-way products

3-way products with a repeated factor of the form $\mathbf{YXX}$ can be used as **formula factories**.

Example: $\mathbf{m} = \mathbf{m}(\mathbf{nn}) = (\mathbf{mn})\mathbf{n}$ since for a proper line $\mathbf{n}^2 = 1$ and associativity. This leads to a decomposition of $\mathbf{m}$ with respect to $\mathbf{n}$:

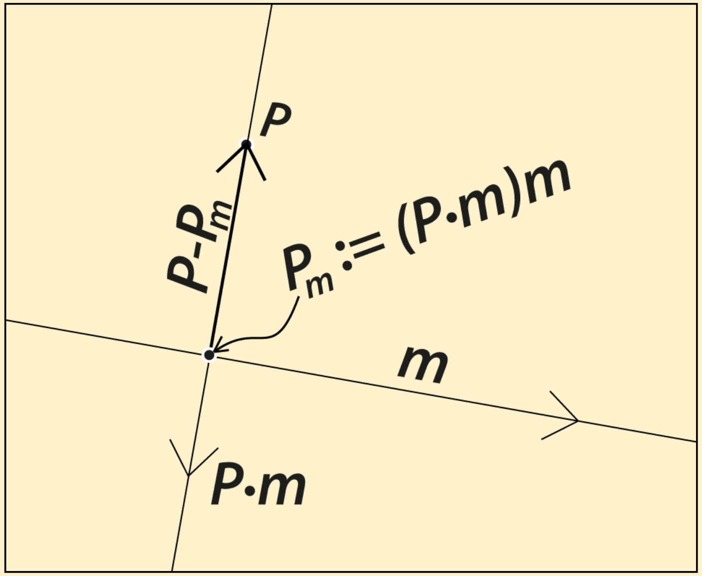
$$
\begin{aligned}
(mn)n &= (\cos(\alpha) + \sin(\alpha)\mathbf{P})n \\
&= \cos(\alpha)n + \sin(\alpha)Pn \\
&= \cos(\alpha)n + \sin(\alpha)P \cdot n \\
&= \cos(\alpha)n - \sin(\alpha)n \cdot P
\end{aligned}
$$



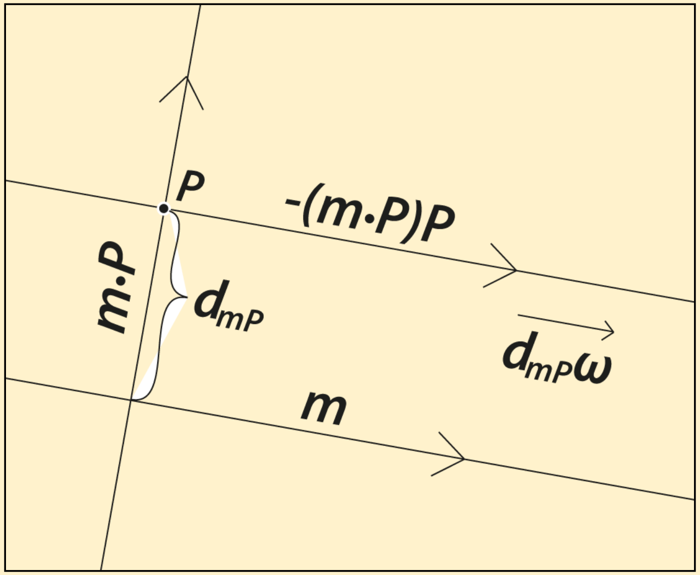
The arrows show the orientation of the lines.

Formula factories via 3-way products

Examples: Anything can be orthogonally decomposed with respect to anything else!   For example ...



Decompose point WRT line.

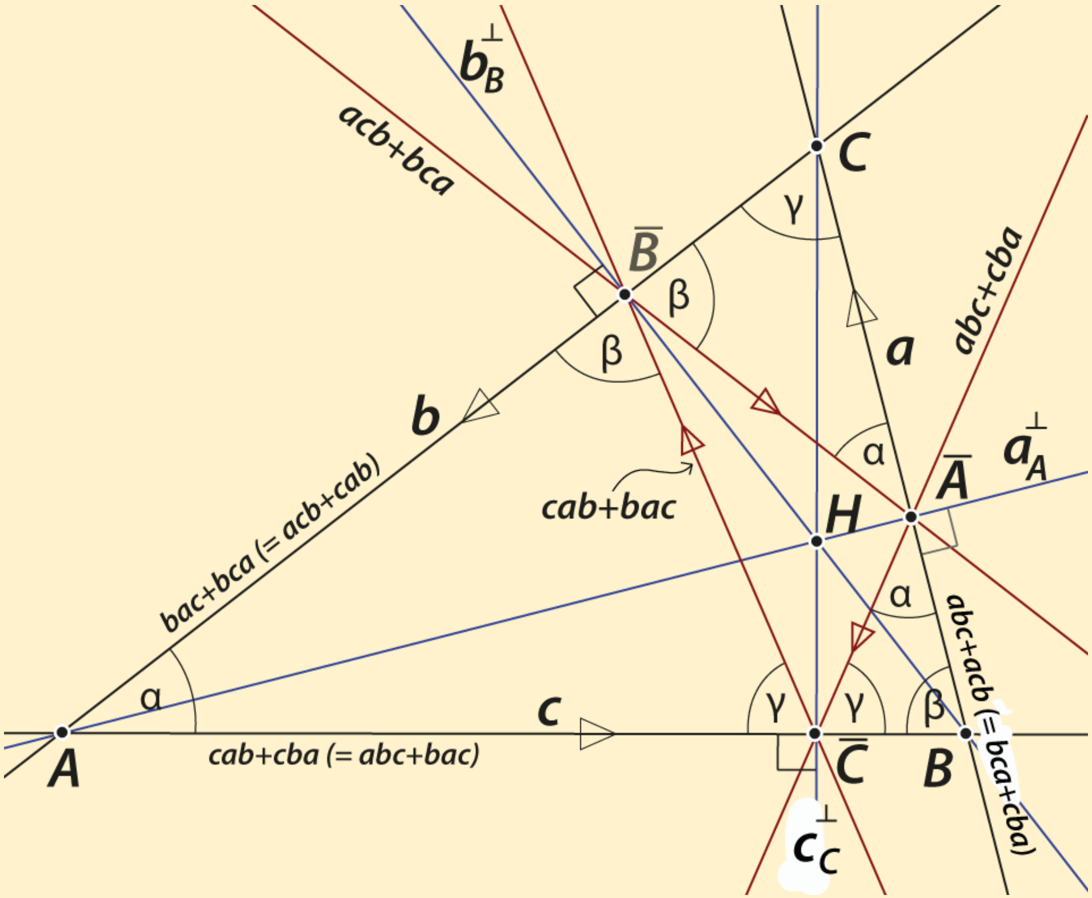


Decompose line WRT point.

The pieces of the decomposition are often interesting in their own right. For example, $(P \cdot m)m$ is closest point to P on m.

Formula factories via 3-way products

General 3-way products $\mathbf{abc}$ of 1-vectors provide a useful framework for a general theory of triangles. Lots left to do!



2D PGA in the browser

A euclidean demo from Steven De Keninck, using his ganja.js Javascript implementation, showing several of the features discussed above.

https://enkimute.github.io/ganja.js/examples/coffeeshop.html#iAdRREx-M&fullscreen

ℓ = line (vector)

P = point (bivector)

ℓP = line through P, $\perp$ to ℓ

$\ell P \ell$ = reflection of P in ℓ

$P \ell P$ = reflection of ℓ in P

$(\ell \cdot P)\ell$ = projection of P on ℓ

$(P \cdot \ell)P$ = projection of ℓ on P

Glimpse at 3D

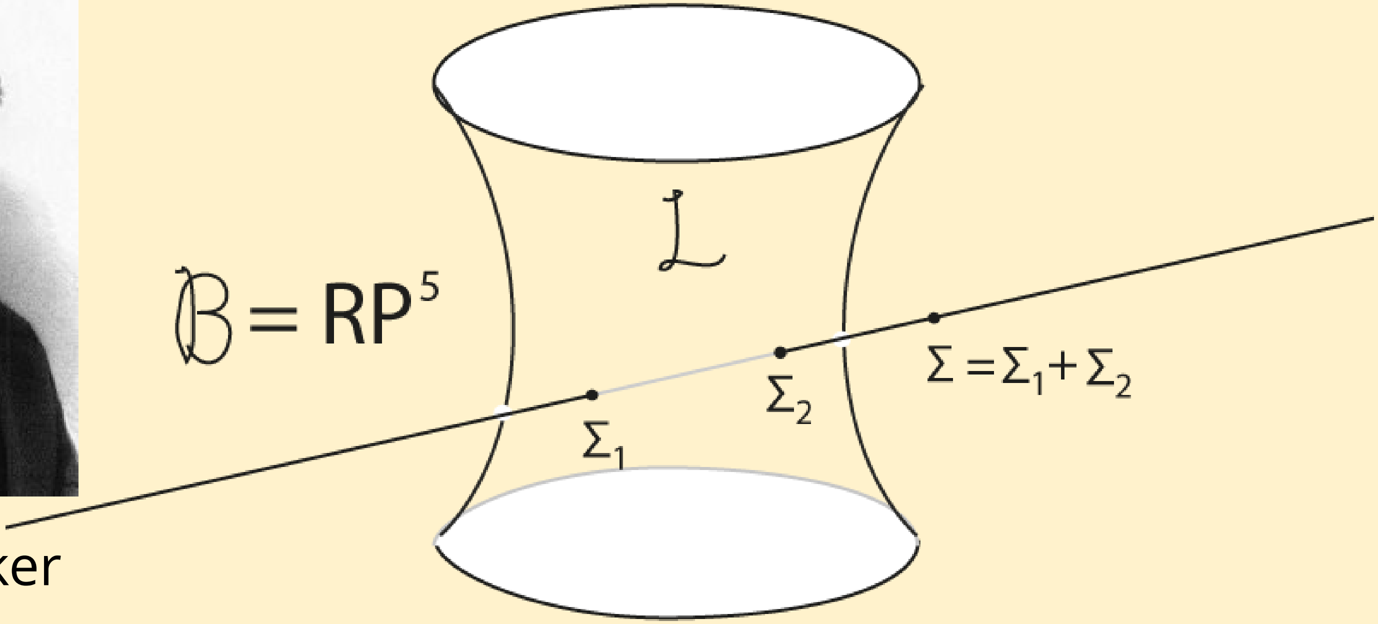
Bivectors!

Julius Pluecker

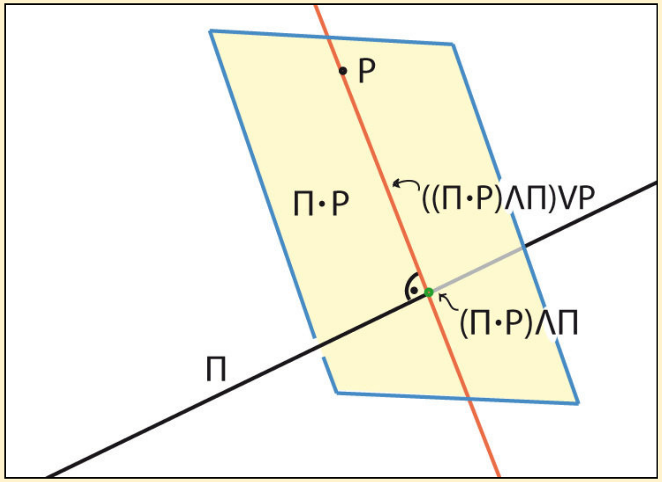
Glimpse at 3D

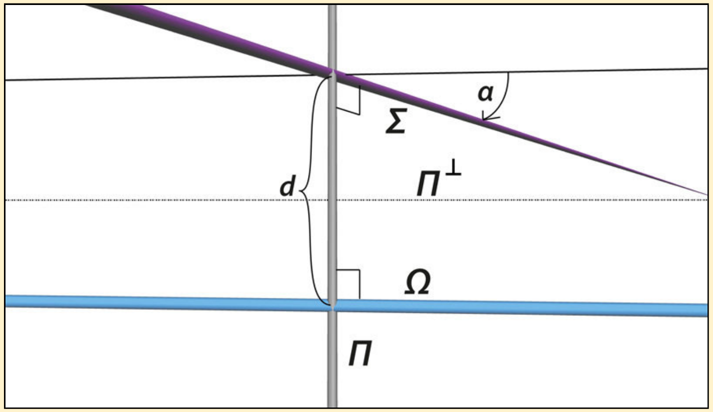
Bivectors!

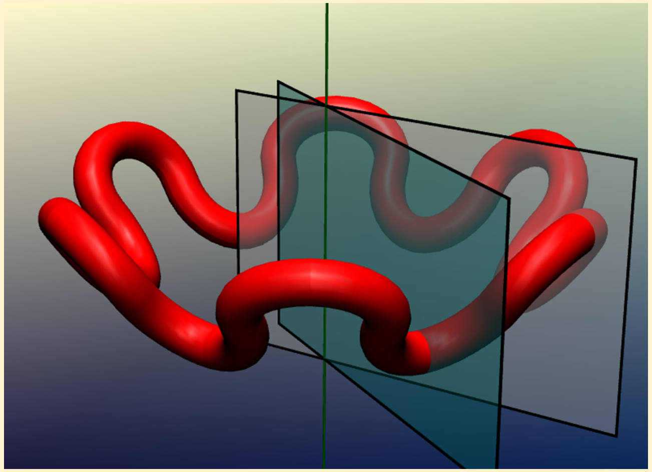
Julius Pluecker

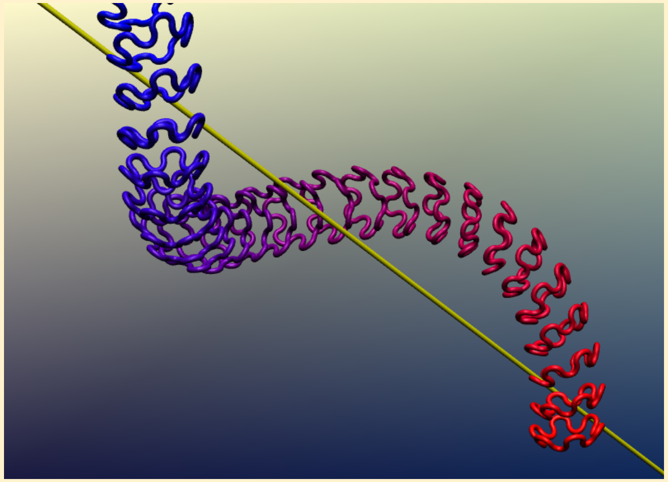


Glimpse at 3D









Kinematics and Mechanics

A velocity state is $\mathbf{V} \in \bigwedge^2$ (in this case a point)

A momentum state is $\mathbf{M} \in \bigwedge^{n-2}$ (in this case a line)

A rigid body is a collection of Newtonian mass points.

Calculate inertia tensor A for the body, a quadratic form determined by the mass distribution.

$\mathbf{M} = A\mathbf{V}$

ODE's for free top:

PGA equations for the free top in $P(\mathbb{R}^*_\kappa)$:

$$\dot{\mathbf{g}} = \mathbf{g}\mathbf{V}$$
$$\dot{\mathbf{M}} = \frac{1}{2}(\mathbf{V}\mathbf{M} - \mathbf{M}\mathbf{V})$$

where $\mathbf{g} \in \mathbf{G}$, and $\mathbf{M}$ and $\mathbf{V}$are in the body frame.

2D Kinematics and Mechanics

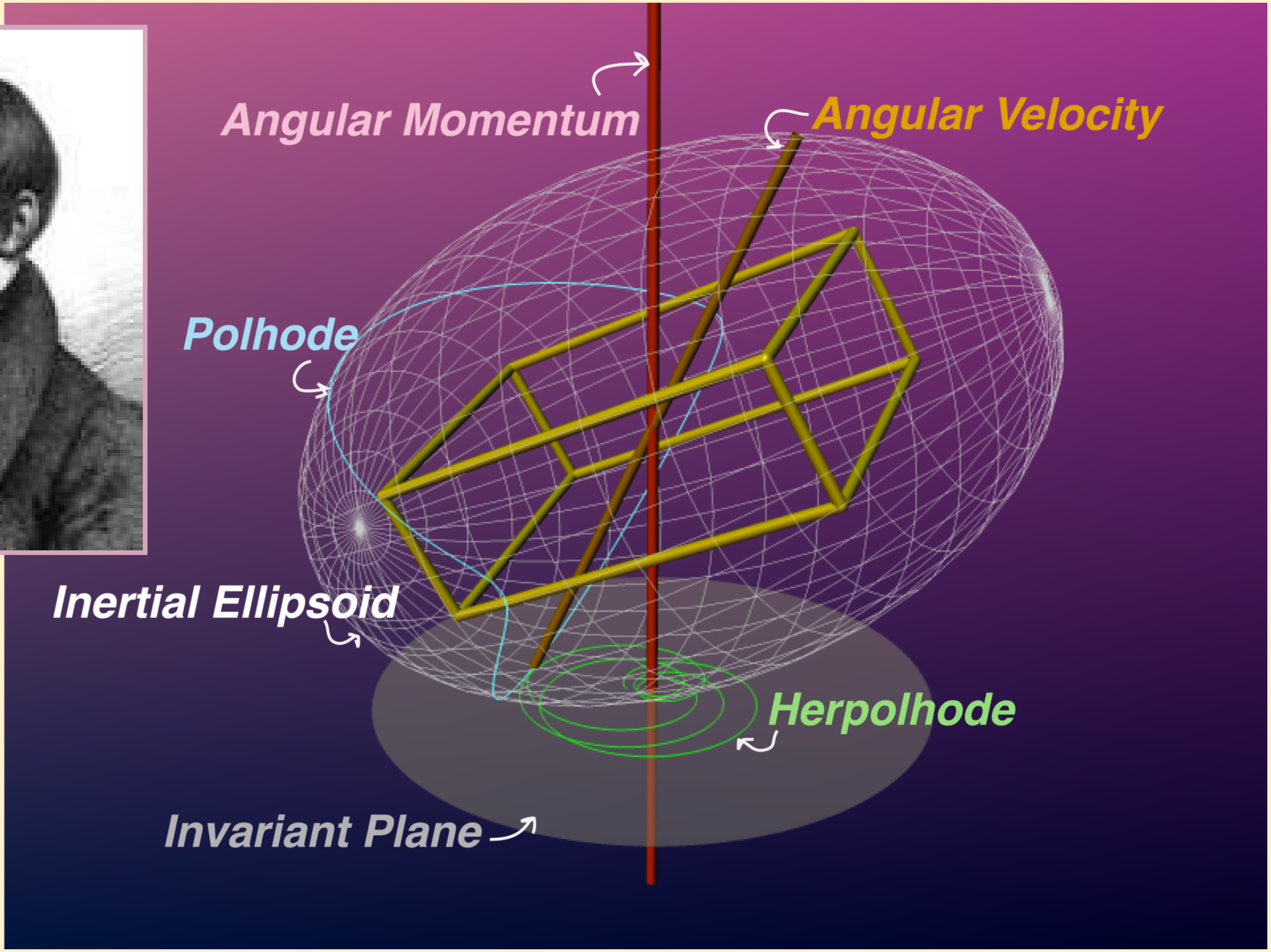
Advantages of PGA:

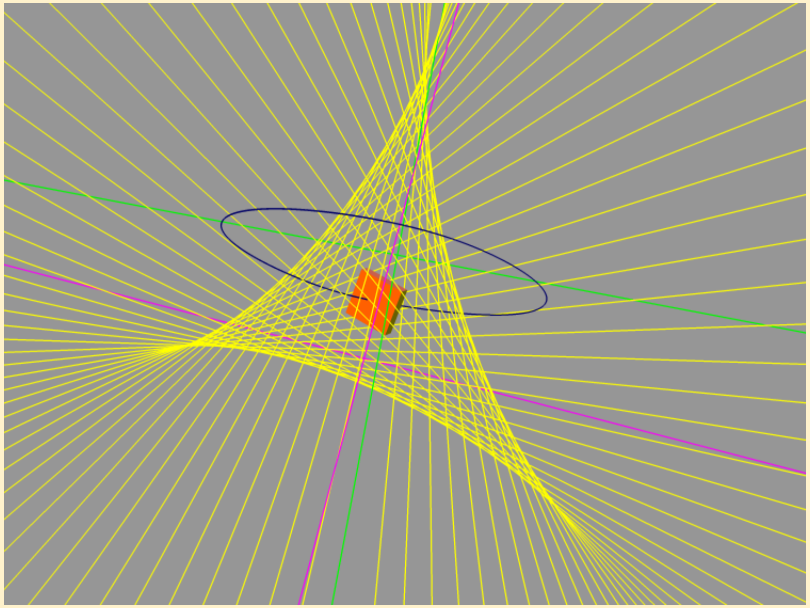
1. **Euclidean case**: No splitting into linear and angular parts. A linear velocity is a velocity carried by an ideal point (euclidean). An angular momentum (or force couple) is one carried by the ideal line.
2. Similar results hold in 3D.
3. The equations are numerically optimal compared to matrix methods. Normalizing $\mathbf{g}$ keeps it on the solution space.

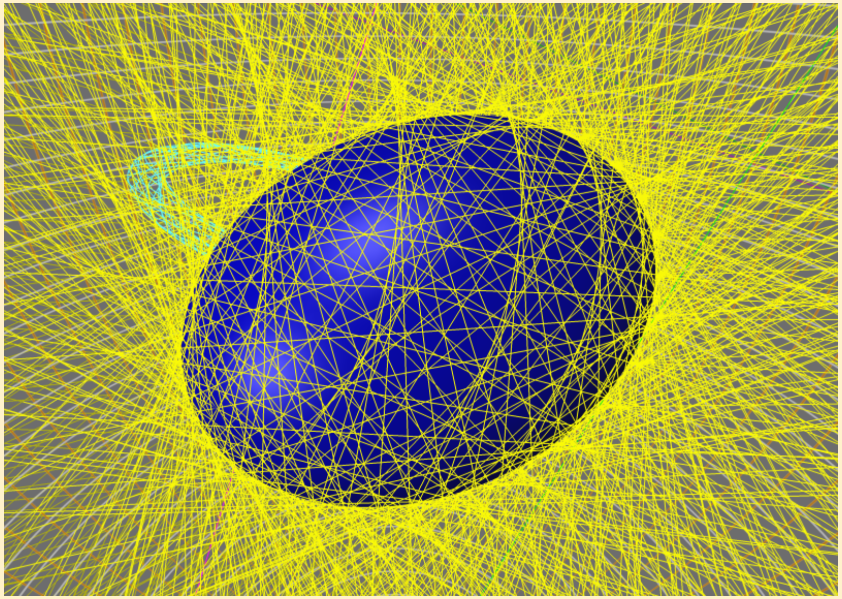
2D Kinematics and Mechanics

https://player.vimeo.com/video/358743032?api=1

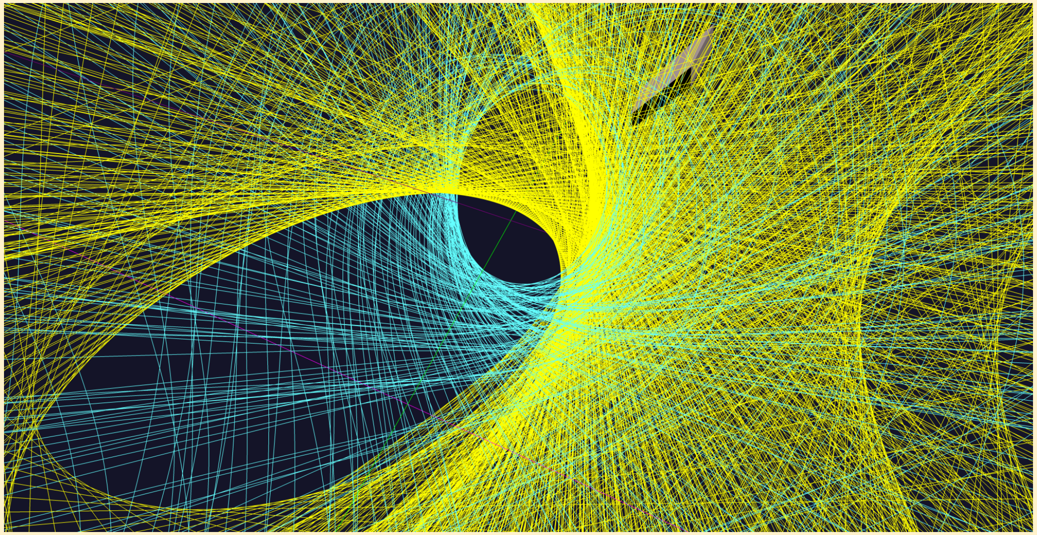
3D Kinematics and Mechanics







3D Poinsot motion (?)



Cayley-Klein programmer's wish list

Cayley-Klein programmer's wish list

Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Cayley-Klein programmer's wish list

Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Metric-neutral

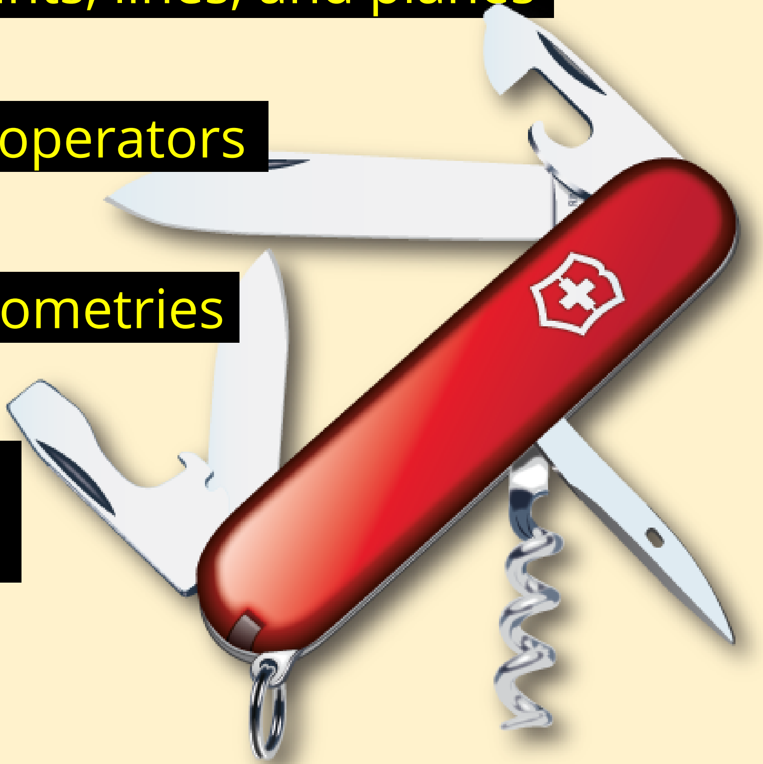
Cayley-Klein programmer's wish list

Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Single, uniform rep'n for isometries

Compact expressions for classical geometric results



Metric-neutral

Cayley-Klein programmer's wish list

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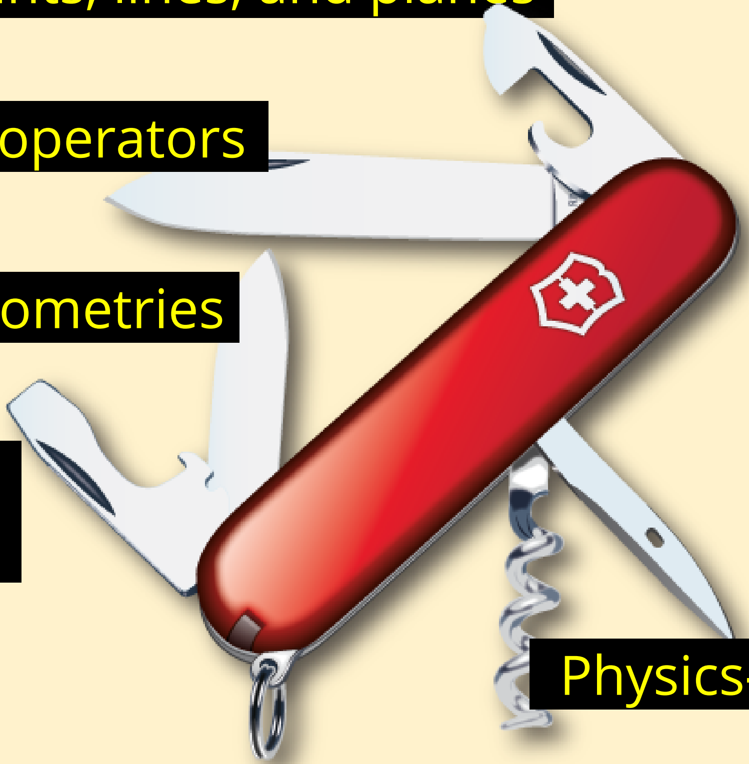

Physics-ready

Metric-neutral

Cayley-Klein programmer's wish list

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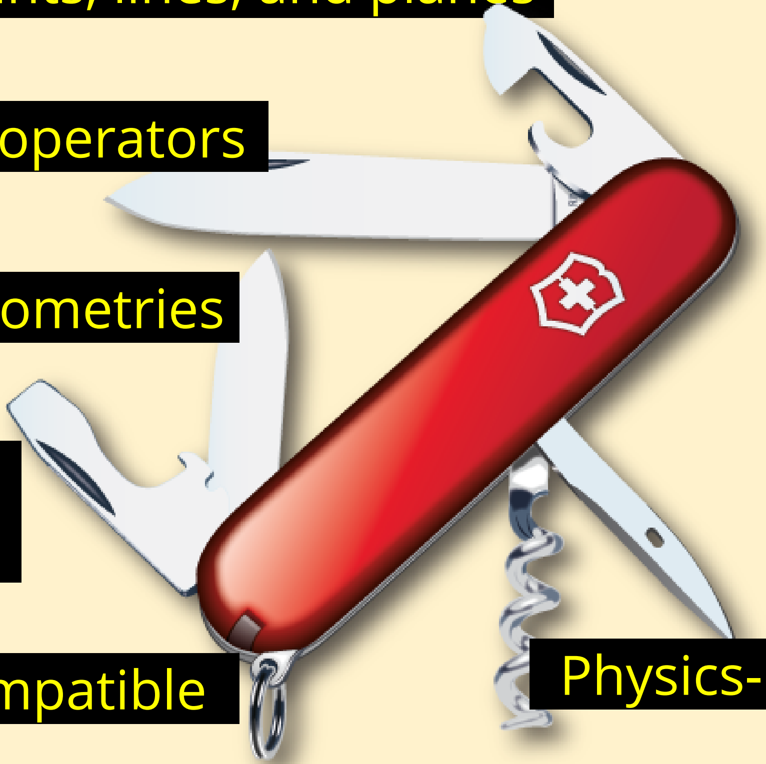


Backwards compatible

Physics-ready

Metric-neutral

Cayley-Klein programmer's wish list

Uniform rep'n for points, lines, and planes

Parallel-safe meet and join operators

Single, uniform rep'n for isometries

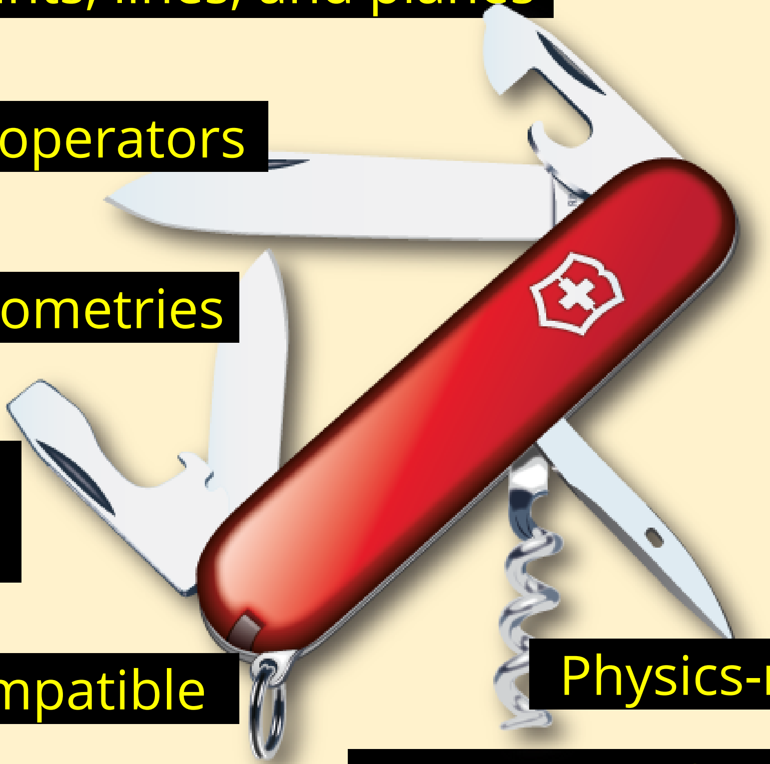
Compact expressions for classical geometric results

Backwards compatible

Metric-neutral

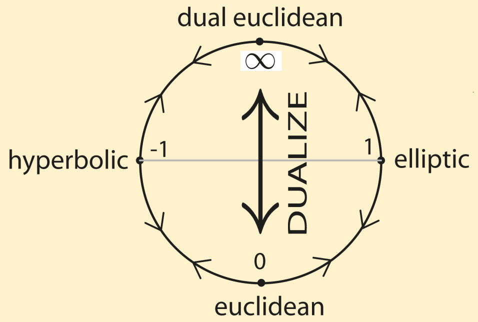
Coordinate-free

Physics-ready



Conclusions

- **Dual PGA** fulfills the "programmers wish list" from the beginning.
- It completes Clifford's project (cut short by his death) of combining Grassmann algebra with all biquaternions, not just the elliptic ones.
- There's a lot left to explore, both in non-euclidean and euclidean, 2D and 3D.
- Team members sought to create browser-based metric-neutral PGA scene graph with physics engine.
- Ask me about **ideal norms** and **dual euclidean space.**



Resources

Javascript implementation

Steven De Keninck

ganja.js

{ 2D Projective Geometric Algebra }

2D PGA Cheat Sheet   SIGGRAPH 2019 Course Notes

Basics

Basis & Metric:

$$\mathbb{R}^*_{2,0,1}$$

	Vector			Bivector			I= PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
+1	0	+1	+1	0	0	−1	0
	LINE : $\boldsymbol{\ell}$			POINT : P			

Multiplication Table:

1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
$\mathbf{e}_0$	0	$\mathbf{e}_{01}$	$-\mathbf{e}_{20}$	0	0	$\mathbf{e}_{012}$	0
$\mathbf{e}_1$	$-\mathbf{e}_{01}$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_0$	$\mathbf{e}_{012}$	$\mathbf{e}_2$	$\mathbf{e}_{20}$
$\mathbf{e}_2$	$\mathbf{e}_{20}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{012}$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$\mathbf{e}_{01}$
$\mathbf{e}_{01}$	0	$\mathbf{e}_0$	$\mathbf{e}_{012}$	0	0	$-\mathbf{e}_{20}$	0
$\mathbf{e}_{20}$	0	$\mathbf{e}_{012}$	$-\mathbf{e}_0$	0	0	$\mathbf{e}_{01}$	0
$\mathbf{e}_{12}$	$\mathbf{e}_{012}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{20}$	$-\mathbf{e}_{01}$	-1	$-\mathbf{e}_0$
$\mathbf{e}_{012}$	0	$\mathbf{e}_{20}$	$\mathbf{e}_{01}$	0	0	$-\mathbf{e}_0$	0

Operators:

$\mathbf{ab}$		Geometric Product	
$\mathbf{a}^*$		Dual	
$\mathbf{a}^\perp$	$\mathbf{a}\mathbf{I}$	Polar	
$\tilde{\mathbf{a}}$		Reverse	
$\langle\mathbf{a}\rangle_n$		Select grade n	
$\mathbf{a}\wedge\mathbf{b}$	$\langle\mathbf{ab}\rangle_{s+t}$	Outer Product	meet
$\mathbf{a}\vee\mathbf{b}$	$(\mathbf{a}^*\wedge\mathbf{b}^*)^*$	Regressive Product	join
$\mathbf{a}\cdot\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\lvert s-t\rvert}$	Inner Product	
$\mathbf{a}\times\mathbf{b}$	$\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$	Commutator Product	
	$\mathbf{a}\mathbf{b}\tilde{\mathbf{a}}$	Sandwich Product	

Dual, Reverse:

Multivector	$a+b\mathbf{e_0}+c\mathbf{e_1}+d\mathbf{e_2}+e\mathbf{e_{01}}+f\mathbf{e_{20}}+g\mathbf{e_{12}}+h\mathbf{e_{012}}$
Dual	$h+g\mathbf{e}^0+f\mathbf{e}^1+e\mathbf{e}^2+d\mathbf{e}^{01}+c\mathbf{e}^{20}+b\mathbf{e}^{12}+a\mathbf{e}^{012}$
Reverse	$a+b\mathbf{e_0}+c\mathbf{e_1}+d\mathbf{e_2}-e\mathbf{e_{01}}-f\mathbf{e_{20}}-g\mathbf{e_{12}}-h\mathbf{e_{012}}$

Sub-algebras:

$\{1\}$	$\mathbb{R}$	Real	$\{1,\mathbf{e_{12}}\}$	$\mathbb{C}$	Complex
$\{1,\mathbf{e_0}\}$	$\mathbb{D}$	Dual	$\{1,\mathbf{e_1}\}$	$\mathbb{D}$	Hyperbolic
$\{1,\mathbf{e_{12}}\}$		rotors	$\{1,\mathbf{e_{01}},\mathbf{e_{20}}\}$		translators
$\{1,\mathbf{e_{01}},\mathbf{e_{20}},\mathbf{e_{12}}\}$		motors			

Geometry

Points, Lines:

Euclidean point at (x,y)	$x\mathbf{e}_{20}+y\mathbf{e}_{01}+\mathbf{e}_{12}$
Direction (ideal point) (x,y)	$x\mathbf{e}_{20}+y\mathbf{e}_{01}$
Line with eq. $a\mathbf{x}+b\mathbf{y}+c=0$	$\boldsymbol{\ell}=a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_0$

Incidence:

Join points $\mathbf{P}_1,\mathbf{P}_2$ in line $\boldsymbol{\ell}$	$\boldsymbol{\ell}=\mathbf{P}_1\vee\mathbf{P}_2$
Meet lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$ in point P	$P=\boldsymbol{\ell}_1\wedge\boldsymbol{\ell}_2$

Project, Reject:

Line orthogonal to line $\boldsymbol{\ell}$, through point P	$\boldsymbol{\ell}\cdot\mathbf{P}=\boldsymbol{\ell}\times\mathbf{P}$
Project point $\mathbf{P}$ on line $\boldsymbol{\ell}$	$(\boldsymbol{\ell}\cdot\mathbf{P})\boldsymbol{\ell}$
Project line $\boldsymbol{\ell}$ on point $\mathbf{P}$	$(\boldsymbol{\ell}\cdot\mathbf{P})\mathbf{P}$
Direction orthogonal to line $\boldsymbol{\ell}$	$\boldsymbol{\ell}^\perp:=\boldsymbol{\ell}\mathbf{I}$

Norms and numerical values:

Euc. norm of $\boldsymbol{\ell}=c\mathbf{e}_0+a\mathbf{e}_1+b\mathbf{e}_2$:	$\lVert\boldsymbol{\ell}\rVert:=\sqrt{\boldsymbol{\ell}^2}(=\sqrt{a^2+b^2})$
Euc. norm of $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}+z\mathbf{e}_{12}$:	$\lVert\mathbf{P}\rVert:=\sqrt{\mathbf{P}\tilde{\mathbf{P}}}\ (=\sqrt{z^2})$
Ideal norm of ideal $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}$:	$\lVert\mathbf{P}\rVert_\infty:=\sqrt{x^2+y^2}$
Norm of motor $\mathbf{m}$	$\lVert\mathbf{m}\rVert:=\sqrt{\mathbf{m}\tilde{\mathbf{m}}}$
Numerical value of ideal $\boldsymbol{\ell}=c\mathbf{e}_0$:	$\lVert\boldsymbol{\ell}\rVert_\infty:=c$
Numerical value of pseudoscalar $a\mathbf{I}$	$\lVert a\mathbf{I}\rVert_\infty=a$

Metric:

Distance between points $\mathbf{P}_1,\mathbf{P}_2$	$\lVert\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\rVert$, $\lVert\hat{\mathbf{P}}_1\times\hat{\mathbf{P}}_2\rVert_\infty$
Angle of intersecting lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$	$\cos^{-1}(\hat{\boldsymbol{\ell}}_1\cdot\hat{\boldsymbol{\ell}}_2)$, $\sin^{-1}(\lVert\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\rVert)$
Distance parallel lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$	$\lVert\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\rVert_\infty$
Oriented dist. eucl. $\mathbf{P}$ to line $\boldsymbol{\ell}$	$\hat{\mathbf{P}}\vee\hat{\boldsymbol{\ell}}$, $\lVert\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\rVert_\infty$
Angle betw. ideal $\mathbf{P}$ and line $\boldsymbol{\ell}$	$\sin^{-1}\lVert\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\rVert_\infty$
Angle bisector of $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$	$(\hat{\boldsymbol{\ell}}_1+\hat{\boldsymbol{\ell}}_2)$ or $\hat{\boldsymbol{\ell}}_1-\hat{\boldsymbol{\ell}}_2$
Perp. bisector of $\mathbf{P}_1$ and $\mathbf{P}_2$	$(\hat{\mathbf{P}}_1+\hat{\mathbf{P}}_2)(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2)$
Altitudes of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$	$(\mathbf{P}_1\vee\mathbf{P}_2)\cdot\mathbf{P}_3$, etc.

Motors

Rotors & Translators:

Rotator α around point $\mathbf{P}_E$	$e^{\frac{\alpha}{2}\mathbf{P}_E}=\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\mathbf{P}_E$
Translator d orthogonal to $\mathbf{P}_\infty$	$e^{\frac{d}{2}\mathbf{P}_\infty}=1+\frac{d}{2}\mathbf{P}_\infty$
Motor between lines $\boldsymbol{\ell}_1$,$\boldsymbol{\ell}_2$	$\sqrt{\hat{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_1}$
Logarithm of motor $\mathbf{m}$	$\widehat{\langle\mathbf{m}\rangle_2}$

Compose & Apply:

Compose motors $\mathbf{m}_1$ and $\mathbf{m}_2$	$\mathbf{m}_2\mathbf{m}_1$
Normalize motor $\mathbf{m}$	$\hat{\mathbf{m}}=\dfrac{\mathbf{m}}{\lVert\mathbf{m}\rVert}$
Square root of motor $\mathbf{m}$	$\sqrt{\mathbf{m}}=(1+\hat{\mathbf{m}})$
Reflect element $\mathbf{X}$ in line $\boldsymbol{\ell}$	$\boldsymbol{\ell}\mathbf{X}\boldsymbol{\ell}$
Transform $\mathbf{X}$ with motor $\mathbf{m}$	$\mathbf{m}\mathbf{X}\tilde{\mathbf{m}}$

More

Areas:

Area of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$	$\frac{1}{2}(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\vee\hat{\mathbf{P}}_3)$
Length of closed loop $\mathbf{P}_1\mathbf{P}_2...\mathbf{P}_n$	$\frac{1}{2}\sum_{i=1}^{n-1}\lVert\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1}\rVert$
Area of closed loop $\mathbf{P}_1\mathbf{P}_2...\mathbf{P}_n$	$\frac{1}{2}\lVert(\sum_{i=1}^{n-1}\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1})\rVert_\infty$

Rigid Body Mechanics: (Valid in euclidean, elliptic & hyperbolic planes)

Kinematics-points, dynamics-lines	linear+angular unified
Element in the body/space frame	$\mathbf{x}_b/\mathbf{x}_s$
Path of $\mathbf{x}$ under the motion $\mathbf{g}$	$\mathbf{x}_s=\mathbf{g}\mathbf{x}_b\tilde{\mathbf{g}}$, $\mathbf{x}_b=\tilde{\mathbf{g}}\mathbf{x}_s\mathbf{g}$
Velocity $\mathbf{V}_b$ in the body	$\mathbf{V}_b=\tilde{\mathbf{g}}\dot{\mathbf{g}}$ (a bivector)
Inertia tensor $\mathsf{A}:\bigwedge^2\leftrightarrow\bigwedge^1$	maps vel. $\leftrightarrow$ mom. in body
Momentum line $\mathbf{m}_b$ in the body	$\mathbf{m}_b=\mathsf{A}(\mathbf{V}_b)$
Kinetic energy E	$E=\mathbf{m}_b\vee\mathbf{V}_b$
Euler Eq. of Motion 1:	$\dot{\mathbf{g}}=\mathbf{g}\mathbf{V}_b$
Euler EoM 2: ($\mathbf{f}_b$ = ext. forces)	$\dot{\mathbf{V}}_b=2\mathsf{A}^{-1}(\mathbf{f}_b+(\mathbf{m}_b\times\mathbf{V}_b))$
Time derivative of energy E	$\dot{E}=-2\mathbf{f}_b\vee\mathbf{V}_b$
Work $w(t)=E(t)-E(0)$	$=\int_0^t\dot{E}ds=-2\int_0^t\mathbf{f}_b\vee\mathbf{V}_b ds$

Resources

Metric-neutral resources

- My Ph. D. thesis
- ganja.js

Euclidean resources

- bivector.net/doc SIGGRAPH 2019 course notes & cheat sheets & course videos + more.
- Live 2D and 3D PGA demos in JavaScript
- My ResearchGate PGA project

Questions and comments: projgeom at gmail.com

Thanks for your attention!

That's all folks

Partial solutions: Quaternions

Quaternions $\mathbb{H}$	$s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Im. quaternions $\mathbb{IH}$	$\mathbf{v} := x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \;\Leftrightarrow (x, y, z) \in \mathbb{R}^3$
Unit quaternions $\mathbb{U}$	$\{\mathbf{g} \in \mathbb{H} \mid \mathbf{g}\overline{\mathbf{g}} = 1\}$

III. ODE's for Euler top:

Quaternion equations for the Euler top in $\mathbb{R}^3$:

$$\dot{\mathbf{g}} = \mathbf{g}\mathbf{V}$$
$$\dot{\mathbf{M}} = \frac{1}{2}(\mathbf{V}\mathbf{M} - \mathbf{M}\mathbf{V})$$

where $\mathbf{g} \in \mathbb{U}$ and $\mathbf{M}, \mathbf{V} \in \mathbb{IH}$ are the momentum, resp., velocity vectors in the body frame.

($\mathbf{M} = A\mathbf{V}$ for inertia tensor A).

Dual projective Grassmann algebra

	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{e}_0$	$\mathbf{e}_0$		$\mathbf{E}_2$	$-\mathbf{E}_1$	$\mathbf{I}$			
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{E}_2$		$\mathbf{E}_0$		$\mathbf{I}$		
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{E}_1$	$-\mathbf{E}_0$				$\mathbf{I}$	
$\mathbf{E}_0$	$\mathbf{E}_0$	$\mathbf{I}$						
$\mathbf{E}_1$	$\mathbf{E}_1$		$\mathbf{I}$					
$\mathbf{E}_2$	$\mathbf{E}_2$			$\mathbf{I}$				
$\mathbf{I}$	$\mathbf{I}$							

Multiplication table for $\bigwedge \mathbb{R}P^{2*}$

Geometric algebra notation

- General *multivector* is sum of k-vectors: $\mathbf{a} = \mathbf{\Sigma}_k \langle \mathbf{a} \rangle_k$
- Points are large letters ($\mathbf{P}$) and lines are small ($\mathbf{m}$).
- The unit pseudoscalar is written $\mathbf{I}$.
- The product of a *k*-vector and an *m*-vector is a sum

  $$\mathbf{KM} = \mathbf{\Sigma}_{i=|k-m|}^{k+m} \langle \mathbf{KM} \rangle_i$$

  where i increases by steps of 2.
- $\mathbf{K} \wedge \mathbf{M} = \langle \mathbf{KM} \rangle_{k+m}$
- $\mathbf{K} \cdot \mathbf{M} := \langle \mathbf{KM} \rangle_{|k-m|}$
- $\mathbf{K} \times \mathbf{M} := \mathbf{KM} - \mathbf{MK}$
- $\mathbf{K} \vee \mathbf{M}$ is the join.

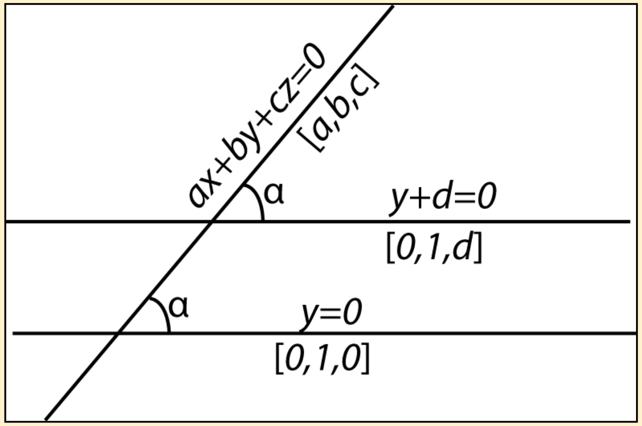
The euclidean algebra $\mathbf{P}(\mathbb{R}^*_{2,0,1})$

Question: Why is the signature $(2, 0, 1)$ using the dual construction the proper model for the euclidean plane?

Answer: Given two lines $\mathbf{m}_i = c_i\mathbf{e}_0 + a_i\mathbf{e}_1 + b_i\mathbf{e}_2$ (with equations $a_i x + b_i y + c_i = 0$). Then

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = c_0 c_1 \mathbf{e}_0^2 + a_1 a_2 \mathbf{e}_1^2 + b_1 b_2 \mathbf{e}_2^2$$

Since the cosine of the angle between the lines is $a_1 a_2 + b_1 b_2$, $\mathbf{e}_0^2 = 0$ while $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$.



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History: That $\mathbf{P}(\mathbb{R}^*_{2,0,1})$ models euclidean geometry was first published by Jon Selig in 2000.

Question

What is the best way
to do Cayley-Klein geometry
on the computer?